Lecture Notes for Math M721

Sheaves and Motivic Homotopy Theory

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Fall 2026

Compiled: September 17, 2026

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Warning. These notes are a work in progress! Everything is subject to change. The source code for these lecture notes is hosted on GitHub. You can submit errata via GitHub Issues.

Disclaimer. A large language model has been/may be used to aid in tasks such as the following:

- Identifying typos and grammatical errors.
- Writing TeX macros and environments.
- Producing TikZ diagrams.

All contentful text in this document is written by myself.

Conventions and Notation

(CN1) I will use $\subset$ or $\subsetneq$ to denote proper subsets, and $\subseteq$ to denote not-necessarily-proper subsets. Likewise for $<$ and $\leq$, $\triangleleft$ and $\trianglelefteq$, etc.

(CN2) I will use $-$ for set difference, as opposed to $\setminus$.

(CN3) Specific categories will be written with sans-serif fonts, e.g. Set is the category of sets. Generic (i.e. quantified or unbound) categories will be written as $\mathcal{C}$, $\mathcal{D}$, etc.

(CN4) If X and Y are two objects in a category $\mathcal{C}$, I may write any of $\mathcal{C}(X, Y)$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, or $\mathrm{Hom}(X, Y)$ to denote the hom-set (or more generally, hom-object) of morphisms from X to Y in $\mathcal{C}$.

(CN5) pt will denote the one-point set $\{\varnothing\}$, or the one-point space with underlying set $\{\varnothing\}$, or the one-point smooth manifold with underlying set $\{\varnothing\}$, etc. etc., depending on context.

Acknowledgements

A hearty thanks is due to the participants in M721.

Chapter 1

Sheaves

1.1 Introduction

Approx. Start of Lecture 1, Aug 25, 2026

Functions[1] are arguably the most fundamental concept in mathematics. A key feature of functions is that they may be *restricted*: given a function $f : X \to Y$ and a subset $X' \subseteq X$, we can produce a function

$$f|_{X'} : X' \to Y$$

defined simply by $f|_{X'}(x) = f(x)$. Equivalently, $f|_{X'}$ is the composite

$$X' \hookrightarrow X \xrightarrow{f} Y.$$

Of course, unless $X' = X$, the restriction $f|_{X'}$ holds less information than f – it no longer remembers the behavior of f on the complement $X - X'$. However, the pair of functions $(f|_{X'}, f|_{X-X'})$ *is* enough to recover the entire data of f; indeed we have

$$f(x) = \begin{cases} f|_{X'}(x) & : x \in X' \\ f|_{X-X'}(x) & : x \notin X' \end{cases}$$

for all $x \in X$. There is no more data in f than what it does on X' and its complement $X - X'$. This is a special case of the following more general slogan.

Slogan 1. Functions are globally determined by their local behavior.

This is only a slogan, so the terms "global" and "local" (and even "function") here are meant to be interpreted loosely. The central aim of this chapter is to develop the understanding that functions are *precisely* those things which are globally determined by their local behavior. The desired understanding is

[1] For now, in this introduction, we mean simply functions between sets, i.e. morphisms in Set.

unrigorous, i.e. we aim to develop a certain mathematical intuition for what it means to be "globally determined by local behavior", and an intuition for what it means that functions are precisely those mathematical objects with this property. As with any goal of this flavor, we must develop the desired intuition by instantiating it in precise theorems.

We can summarize our first example by noticing we have an isomorphism

$$\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(X', Y) \times \mathrm{Hom}(X - X', Y)$$

for any sets X and Y and any subset $X' \subseteq X$. This is a special case of the universal property of coproducts, and more importantly for our purposes, it is also a special case of a *sheaf condition*. We proceed by analyzing a more general situation.

Let X' and X'' be subsets of X. Then, given a function $f : X \to Y$, we may form the restrictions $f|_{X'}$ and $f|_{X''}$. This gives a map

$$\begin{aligned} \mathrm{Hom}(X, Y) &\to \mathrm{Hom}(X', Y) \times \mathrm{Hom}(X'', Y) \\ f &\mapsto (f|_{X'}, f|_{X''}) \end{aligned}$$

which in general is neither injective nor surjective. However, it is easy to understand precisely the failures of both of these conditions.

Proposition 1.1.1. *The map*

$$\mathrm{Hom}(X, Y) \to \mathrm{Hom}(X', Y) \times \mathrm{Hom}(X'', Y)$$

is injective for all sets Y if and only if $X' \cup X'' = X$.

Proof. Elementary. □

Note that this map being injective is the same as saying that f is always reconstructible from the data $(f|_{X'}, f|_{X''})$.

Proposition 1.1.2. *When $X = X' \cup X''$, the image of the map*

$$\mathrm{Hom}(X, Y) \to \mathrm{Hom}(X', Y) \times \mathrm{Hom}(X'', Y)$$

is precisely the set

$$\{(f', f'') \in \mathrm{Hom}(X', Y) \times \mathrm{Hom}(X'', Y) : f'|_{X' \cap X''} = f''|_{X' \cap X''}\}.$$

Proof. Elementary. □

To summarize all of the above: if X' and X'' together cover X, then the data of a function on X is equivalent to a pair of functions (one on X' and one on X'')

which agree on the overlap $X' \cap X''$. In other words, whenever $X = X' \cup X''$, we have a pullback square

$$\begin{tikzcd}
\mathrm{Hom}(X,Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Hom}(X',Y) \arrow[d] \\
\mathrm{Hom}(X'',Y) \arrow[r] & \mathrm{Hom}(X' \cap X'',Y)
\end{tikzcd}$$

in the category of sets.

The situation is very much the same if we cover X by more than two pieces.

Proposition 1.1.3. *Let $X = \bigcup_\alpha X_\alpha$ for some collection of sets $\{X_\alpha\}_\alpha$. Then for any set Y, there is an equalizer diagram*

$$\mathrm{Hom}(X,Y) \to \prod_\alpha \mathrm{Hom}(X_\alpha, Y) \rightrightarrows \prod_{\alpha,\beta} \mathrm{Hom}(X_\alpha \cap X_\beta, Y)$$

in the category of sets. Here the two parallel arrows are defined by

$$(f_\alpha)_\alpha \mapsto (f_\alpha|_{X_\alpha \cap X_\beta})_{\alpha,\beta}$$

and

$$(f_\alpha)_\alpha \mapsto (f_\beta|_{X_\alpha \cap X_\beta})_{\alpha,\beta}.$$

Proof. Exercise. In more detail:

1. Make it clear to yourself exactly what this proposition is saying.
2. Make it clear to yourself why this generalizes the previous discussion.
3. Prove the proposition.
4. Remembering our earlier slogan, you should think to yourself about how this says that "functions on X are determined globally by their local behavior".

$\square$

We finish with an instantiation of our main goal: to understand that functions are *precisely* those things which are determined globally by their local behavior.

Theorem 1.1.4. *Let $\mathcal{F} : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$ be a functor satisfying the following "sheaf condition":*

($\star$) *For any set X and any collection of subsets $\{X_\alpha\}_\alpha$ of X, if $X = \bigcup_\alpha X_\alpha$ then*

$$\mathcal{F}(X) \to \prod_\alpha \mathcal{F}(X_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathcal{F}(X_\alpha \cap X_\beta)$$

is an equalizer diagram in Set, where the maps above are given by applying $\mathcal{F}$ to the inclusions (as in Proposition 1.1.3).

Then $\mathcal{F}$ is representable, i.e. $\mathcal{F} \cong \mathrm{Hom}(-,Y)$ for some set Y.

Proof. Set $Y = \mathcal{F}(\mathrm{pt})$. We first note that $\varnothing = \bigcup \varnothing$, whence we have an equalizer diagram of the form

$$\mathcal{F}(\varnothing) \to \mathrm{pt} \rightrightarrows \mathrm{pt},$$

showing that $\mathcal{F}(\varnothing) \cong \mathrm{pt}$. Here we have used the fact that pt is terminal in Set (and empty products are terminal objects).

Now let X be an arbitrary set, and write $X = \bigcup_{x \in X} \{x\}$. Then we have an equalizer diagram

$$\mathcal{F}(X) \to \prod_{x \in X} \mathcal{F}(\{x\}) \rightrightarrows \prod_{x,x' \in X} \mathcal{F}(\{x\} \cap \{x'\}).$$

We note that $\mathcal{F}(\{x\} \cap \{x'\}) = \mathcal{F}(\varnothing) \cong \mathrm{pt}$ unless $x = x'$. Therefore we have an equalizer diagram of the form

$$\mathcal{F}(X) \to \prod_{x \in X} \mathcal{F}(\{x\}) \rightrightarrows \prod_{x \in X} \mathcal{F}(\{x\})$$

where the two parallel arrows are equal[a]. We conclude that the map

$$\mathcal{F}(X) \to \prod_{x \in X} \mathcal{F}(\{x\})$$

is an isomorphism. This allows us to construct an element

$$i \in \mathcal{F}(Y) \cong \prod_{y \in Y} \mathcal{F}(\{y\}) \cong \prod_{y \in Y} \mathcal{F}(\mathrm{pt}) = \prod_{y \in Y} Y$$

from the tuple

$$(y)_{y \in Y},$$

where the isomorphism in the line above is the product of the isomorphisms $\mathcal{F}(\{y\}) \to \mathcal{F}(\mathrm{pt})$ obtained by applying $\mathcal{F}$ to the unique map $\mathrm{pt} \to \{y\}$.

By the Yoneda lemma[b], the element $i \in \mathcal{F}(Y)$ corresponds to a natural transformation $\eta : \mathrm{Hom}(-, Y) \to \mathcal{F}$. Explicitly, η is given by

$$\eta_X(f) = \mathcal{F}(f)(i).$$

We claim that this natural transformation is an isomorphism. For this, let X be an arbitrary set, and notice that we have a commutative[c] diagram

$$\begin{tikzcd}
\mathrm{Hom}(X, Y) \arrow[r, "\cong"] \arrow[d, "\eta_X"'] & \prod_{x \in X} \mathrm{Hom}(\{x\}, Y) \arrow[d, "\prod_{x \in X} \eta_{\{x\}}"] \\
\mathcal{F}(X) \arrow[r, "\cong"'] & \prod_{x \in X} \mathcal{F}(\{x\})
\end{tikzcd}$$

where the top horizontal map is an isomorphism by Proposition 1.1.3. To show that η_X is an isomorphism, it suffices to show that each $\eta_{\{x\}}$ is an isomorphism. We have a naturality square

$$\begin{tikzcd}
\mathrm{Hom}(\{x\}, Y) \arrow[r, "\eta_{\{x\}}"] \arrow[d, "\cong"'] & \mathcal{F}(\{x\}) \arrow[d, "\cong"] \\
\mathrm{Hom}(\mathrm{pt}, Y) \arrow[r, "\eta_{\mathrm{pt}}"'] & \mathcal{F}(\mathrm{pt})
\end{tikzcd}$$

so it suffices to show that $\eta_{\mathrm{pt}} : \mathrm{Hom}(\mathrm{pt}, Y) \to \mathcal{F}(\mathrm{pt}) = Y$ is an isomorphism. We first precompose with the isomorphism

$$\begin{aligned} Y &\to \mathrm{Hom}(\mathrm{pt}, Y) \\ y &\mapsto f_y \end{aligned}$$

where $f_y : \mathrm{pt} \to Y$ is the unique function with image $\{y\}$. We will show that the composite $Y \to Y$ is the identity. So, let $y \in Y$ be arbitrary. We have

$$\eta_{\mathrm{pt}}(f_y) = \mathcal{F}(f_y)(i).$$

But by construction of i, we have that[d] $\mathcal{F}(f_y)$ sends i to y. □

[a] Check!
[b] We discuss the Yoneda lemma in detail later, see Proposition 1.2.5.
[c] Check!
[d] Check!

Exercises

Exercise 1.1.1. Prove Propositions 1.1.1 and 1.1.2.

Exercise 1.1.2. Complete the proof of Proposition 1.1.3.

Exercise 1.1.3. Check the details in the proof of Theorem 1.1.4.

1.2 Sheaves on Spaces

Now, let X be a topological space. We are often interested in continuous functions on X. Continuous functions also have the property that they are determined globally by their local behavior: in this context, "local" is best interpreted as "on an open set". That is: a continuous function on X is uniquely determined by its restrictions to the pieces of an open cover of X, and a collection of continuous functions on the pieces of an open cover of X comes from a continuous function on X if and only if the continuous functions agree on pairwise intersections of pieces of the open cover. It is important that we restrict to open subsets here!

For example, consider the functions

$$f_1 : (-\infty, 0) \to \mathbb{R} \qquad\qquad f_2 : [0, \infty) \to \mathbb{R}$$
$$x \mapsto 0 \qquad\qquad x \mapsto 1.$$

We see that f_1 and f_2 are continuous, and we have

$$(-\infty, 0) \cup [0, \infty) = \mathbb{R} \qquad\qquad (-\infty, 0) \cap [0, \infty) = \varnothing,$$

but there is no continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f|_{(-\infty,0)} = f_1$ and $f|_{[0,\infty)} = f_2$.

So, the correct analogue of Proposition 1.1.3 for continuous functions is the following.

Proposition 1.2.1. *Let X and Y be topological spaces, and let $\{U_\alpha\}_\alpha$ be an open cover of X. Then there is an equalizer diagram*

$$\mathrm{Hom}(X, Y) \to \prod_\alpha \mathrm{Hom}(U_\alpha, Y) \rightrightarrows \prod_{\alpha,\beta} \mathrm{Hom}(U_\alpha \cap U_\beta, Y)$$

in the category of sets. Here the two parallel arrows are defined by

$$(f_\alpha)_\alpha \mapsto (f_\alpha|_{U_\alpha \cap U_\beta})_{\alpha,\beta}$$

and

$$(f_\alpha)_\alpha \mapsto (f_\beta|_{U_\alpha \cap U_\beta})_{\alpha,\beta}.$$

Proof. Exercise. □

Approx. Start of Lecture 2, Aug 27, 2026

We note that the analogue of Theorem 1.1.4 does not hold in this context!

Example 1.2.2. Let $q : [0, 1] \to S^1$ be the standard quotient map. We say that a continuous function $g : X \to S^1$ is *locally liftable* along q if, for all $x \in X$, there exists an open neighborhood U of x and a continuous function $\tilde{g} : U \to [0, 1]$ such that $q \circ \tilde{g} = g|_U$.

Let $\mathcal{F} : \mathsf{Top}^{\mathrm{op}} \to \mathsf{Set}$ be the subfunctor[2] of $\mathrm{Hom}(-, S^1)$ defined by

$$\mathcal{F}(X) = \{g \in \mathrm{Hom}(X, S^1) : g \text{ is locally liftable along } q\}.$$

Then the following sheaf condition holds:

($\star$) For any space X and any open cover $\{U_\alpha\}_\alpha$ of X,

$$\mathcal{F}(X) \to \prod_\alpha \mathcal{F}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathcal{F}(U_\alpha \cap U_\beta)$$

is an equalizer diagram in Set.

However, $\mathcal{F}$ is not representable.

[2] This means simply that each $\mathcal{F}(X)$ is a subset of $\mathrm{Hom}(X, S^1)$, and that the inclusions $\mathcal{F}(X) \to \mathrm{Hom}(X, S^1)$ form a natural transformation (this determines what $\mathcal{F}$ does to morphisms). We'll talk more about subfunctors later!

Proof. It is left as an exercise to show that $\mathcal{F}$ is a well-defined subfunctor of $\mathrm{Hom}(-, S^1)$ which satisfies the sheaf condition.

Suppose for contradiction that $\mathcal{F}$ is representable, and pick a space Y such that $\mathcal{F} \cong \mathrm{Hom}(-, Y)$. By the Yoneda lemma, this natural isomorphism is determined by an element $i \in \mathcal{F}(Y)$, i.e. a continuous function $i : Y \to S^1$ which is locally liftable along q. Precisely, the Yoneda lemma tells us that the natural isomorphism $\mathrm{Hom}(-, Y) \to \mathcal{F}$ is given by

$$\begin{aligned} \mathrm{Hom}(X, Y) &\to \mathcal{F}(X) \\ f &\mapsto i \circ f. \end{aligned}$$

The map i also gives us a natural transformation $\mathrm{Hom}(-, Y) \to \mathrm{Hom}(-, S^1)$ by postcomposition. For any space X, we see that the composite

$$\mathcal{F}(X) \xrightarrow{\cong} \mathrm{Hom}(X, Y) \xrightarrow{i_*} \mathrm{Hom}(X, S^1)$$

sends an element $g \in \mathcal{F}(X)$ first to the unique morphism $f \in \mathrm{Hom}(X, Y)$ such that $i \circ f = g$, and then to the morphism $i \circ f = g \in \mathrm{Hom}(X, S^1)$. In other words, this composite is the inclusion $\mathcal{F} \to \mathrm{Hom}(-, S^1)$. In particular, $i_* : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, S^1)$ is injective for all spaces X, i.e. $i : Y \to S^1$ is a monomorphism[a].

Now observe that q is itself locally liftable along q, since $\mathrm{id}_{[0,1]}$ is a global lift. So $q \in \mathcal{F}([0, 1])$, i.e. there is a unique continuous map $f : [0, 1] \to Y$ with $i \circ f = q$. Since q is a quotient map, i is also a quotient map (in particular, i is surjective). We have now established that i is a bijective quotient map, i.e. a homeomorphism. Without loss of generality, $Y = S^1$ and $i = \mathrm{id}_{S^1}$. Then we have $\mathrm{id}_{S^1} = i \in \mathcal{F}(Y) = \mathcal{F}(S^1)$, which is a contradiction: id_{S^1} is not locally liftable along q at the basepoint $* := q(0) = q(1)$. □

[a] In Top, monomorphisms are simply injective continuous functions.

Remark (For experts! This remark can be safely skipped.)**.** There's actually a lot going on in the background with this example. Theorem 1.1.4 is stated in a strange way. The more general fact is that for any site $(\mathcal{C}, \tau)$, we have

$$\mathsf{Sh}(\mathcal{C}, \tau) \simeq \mathsf{Sh}(\mathsf{Sh}(\mathcal{C}, \tau), \mathrm{can})$$

via the Yoneda embedding. Theorem 1.1.4 is an instantiation of this in the case that $\mathcal{C} = *$ is the terminal category (via $\mathsf{Sh}(*) \simeq \mathsf{Set}$), but the statement there is a priori stronger, because we only posit the sheaf condition for literal coverings by subsets $X = \bigcup_\alpha X_\alpha$. This turns out to not make a difference, because the canonical topology on Set is generated by these coverings. So, for Top, there are two reasons to expect an example like Example 1.2.2:

1. Top is not a Grothendieck topos (this is the real sticking point).
2. The canonical topology on Top is not generated by open covers.

We can alleviate a lot of this weirdness by focusing our attention on just one topological space, instead of the entire category Top. First, some theory.

Definition 1.2.3. Let X be a topological space. We denote by $\mathsf{Open}(X)$ the poset of open subsets of X, which we view as a category.

Definition 1.2.4. For any categories $\mathcal{C}$ and $\mathcal{D}$, a *presheaf* on $\mathcal{C}$ valued in $\mathcal{D}$ is a functor $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. The category of presheaves will be denoted $\mathsf{PSh}(\mathcal{C}, \mathcal{D}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$.

A *presheaf* on a space X valued in $\mathcal{D}$ is a functor $\mathcal{F} : \mathsf{Open}(X)^{\mathrm{op}} \to \mathcal{D}$, i.e. $\mathsf{PSh}(X, \mathcal{D})$ is defined to be $\mathsf{PSh}(\mathsf{Open}(X), \mathcal{D})$.

When $\mathcal{D}$ is unspecified, it is assumed to be Set. For example, when X is a space, $\mathsf{PSh}(X)$ denotes the functor category $\mathrm{Fun}(\mathsf{Open}(X)^{\mathrm{op}}, \mathsf{Set})$.

We think of a presheaf $\mathcal{F} \in \mathsf{PSh}(X, \mathcal{D})$ as a way of assigning, to each open subset $U \subseteq X$, a collection of data (an object of $\mathcal{D}$) defined locally on U. As we have already seen, *representable* presheaves are very important. These are intimately connected to the Yoneda lemma and its corollaries.

Proposition 1.2.5 (Yoneda)**.** *Let $\mathcal{C}$ be a small*[3] *category.*

1. *There is a fully faithful functor $y : \mathcal{C} \to \mathsf{PSh}(\mathcal{C})$ (the* Yoneda embedding*) defined on objects by*
$$y(X) = \mathrm{Hom}_{\mathcal{C}}(-, X)$$
and acting on morphisms by postcomposition. Presheaves $\mathcal{F} \in \mathsf{PSh}(\mathcal{C})$ which lie in the essential image of y (i.e. $\mathcal{F} \cong y(X)$ for some X) are called representable.

2. *There is an isomorphism*
$$\mathrm{Hom}_{\mathsf{PSh}(\mathcal{C})}(y(X), \mathcal{F}) \cong \mathcal{F}(X)$$
natural in both X and $\mathcal{F}$. In one direction, a natural transformation $\alpha : y(X) \to \mathcal{F}$ determines an element $\alpha_X(\mathrm{id}_X) \in \mathcal{F}(X)$. In the other direction, an element $x \in \mathcal{F}(X)$ determines a natural transformation $\alpha : y(X) \to \mathcal{F}$ whose components are given by $\alpha_Y(f) = \mathcal{F}(f)(x)$.

3. *y exhibits $\mathsf{PSh}(\mathcal{C})$ as the free cocompletion of $\mathcal{C}$. That is: $\mathsf{PSh}(\mathcal{C})$ is cocomplete, and any functor $F : \mathcal{C} \to \mathcal{D}$ with $\mathcal{D}$ cocomplete factors uniquely (up to natural isomorphism) as $G \circ y$ with $G : \mathsf{PSh}(\mathcal{C}) \to \mathcal{D}$ cocontinuous.*

4. *Every presheaf $\mathcal{F} \in \mathsf{PSh}(\mathcal{C})$ is a colimit (in a canonical way) of representable presheaves. Explicitly, let $\int \mathcal{F}$ (the* category of elements *of $\mathcal{F}$) be the category whose objects are pairs (X, x) where $X \in \mathcal{C}$ and $x \in \mathcal{F}(X)$ – morphisms $(X, x) \to (Y, y)$ in $\int \mathcal{F}$ are morphisms $f : X \to Y$ in $\mathcal{C}$ such that $\mathcal{F}(f)(y) = x$. There is a projection $\pi : \int \mathcal{F} \to \mathcal{C}$ given by $(X, x) \mapsto X$. Composing this projection with the Yoneda embedding gives a*

[3] For now, we will keep as a standing assumption that the domain of any presheaf is small.

functor $y \circ \pi : \int \mathcal{F} \to \mathsf{PSh}(\mathcal{C})$. For each $(X, x) \in \int \mathcal{F}$, there is a natural transformation $(y \circ \pi)(X, x) = y(X) \to \mathcal{F}$ corresponding to the element $x \in \mathcal{F}(X)$. This collection of natural transformations exhibits $\mathcal{F}$ as the colimit of $y \circ \pi$.

Example 1.2.6. Here are some examples of presheaves on spaces.

1. Let X and Y be spaces. There is a presheaf $\mathcal{F} \in \mathsf{PSh}(X)$ given by $\mathcal{F}(U) = \mathrm{Hom}_{\mathsf{Top}}(U, Y)$. This is the composition of the representable presheaf $y(Y) : \mathsf{Top}^{\mathrm{op}} \to \mathsf{Set}$ with the (usually non-full!) inclusion $\mathsf{Open}(X)^{\mathrm{op}} \subseteq \mathsf{Top}^{\mathrm{op}}$.

2. Let X be a space and let Y be a set. There is a presheaf $\mathcal{F} \in \mathsf{PSh}(X)$ given by $\mathcal{F}(U) = Y$ for all $U \in \mathsf{Open}(X)$. This is a constant functor.

3. Let X be a space and let $V \in \mathsf{Open}(X)$. There is a presheaf $\mathcal{F} \in \mathsf{PSh}(X)$ given by
$$\mathcal{F}(U) = \begin{cases} \mathrm{pt} & U \subseteq V, \\ \varnothing & U \not\subseteq V. \end{cases}$$
This is the representable presheaf $y(V)$.

4. Let X be a smooth manifold, so that every open subset of X is also a smooth manifold in a canonical way. There is a presheaf $\mathcal{F} \in \mathsf{PSh}(X)$ given by $\mathcal{F}(U) = \mathrm{Hom}_{\mathsf{Man}}(U, \mathbb{R})$, where Man is the category of smooth manifolds. This is the composition of the representable presheaf $y(\mathbb{R}) : \mathsf{Man}^{\mathrm{op}} \to \mathsf{Set}$ with the (usually non-full!) inclusion $\mathsf{Open}(X)^{\mathrm{op}} \subseteq \mathsf{Man}^{\mathrm{op}}$.

There are many, many more examples, but we must focus on something to start! Now we can turn to the sheaf condition for presheaves on spaces. It is precisely the generalization of the condition in Proposition 1.2.1.

Definition 1.2.7. Let X be a space. A presheaf $\mathcal{F} \in \mathsf{PSh}(X)$ is a *sheaf* if for any open set $U \subseteq X$ and any open cover $\{U_\alpha\}_\alpha$ of U, the diagram
$$\mathcal{F}(U) \to \prod_{\alpha} \mathcal{F}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathcal{F}(U_\alpha \cap U_\beta)$$
is an equalizer diagram in Set. The category of sheaves on X (the full subcategory of $\mathsf{PSh}(X)$ spanned by all sheaves) will be denoted $\mathsf{Sh}(X)$.

Defining sheaves with values in a generic category $\mathcal{D}$ is quite subtle, philosophically. For now, we will handle only the case of "algebraic categories", by which I mean Set, Grp, Ab, Ring, and CRing. In this case, the same definition makes sense, and is "correct".

⚠ **Warning.** We will carry this definition of "algebraic category" throughout this section of the notes. It is extremely nonstandard and should not be taken seriously. The point is just that these particular categories Set, Grp, Ab, Ring, CRing are very well-behaved *and* it is very important to consider sheaves with values in these categories.

Definition 1.2.8. Let X be a space and let $\mathcal{D}$ be an algebraic category. A presheaf $\mathcal{F} \in \mathsf{PSh}(X, \mathcal{D})$ is a *sheaf* if for any open set $U \subseteq X$ and any open cover $\{U_\alpha\}_\alpha$ of U, the diagram

$$\mathcal{F}(U) \to \prod_\alpha \mathcal{F}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathcal{F}(U_\alpha \cap U_\beta)$$

is an equalizer diagram in $\mathcal{D}$. The category of sheaves on X with values in $\mathcal{D}$ (the full subcategory of $\mathsf{PSh}(X, \mathcal{D})$ spanned by all sheaves) will be denoted $\mathsf{Sh}(X, \mathcal{D})$.

Note that, for our algebraic categories (or indeed any category $\mathcal{D}$ which is monadic over Set), the forgetful functor $\mathcal{D} \to \mathsf{Set}$ preserves and creates limits, so a presheaf $\mathcal{F} \in \mathsf{PSh}(X, \mathcal{D})$ is a sheaf if and only if the underlying presheaf of sets is a sheaf. We now turn to the most important example of a sheaf on a space.

Example 1.2.9. Let $\pi : E \to X$ be a continuous function. There is a sheaf $\mathcal{S}_\pi \in \mathsf{Sh}(X)$ defined on objects by

$$\mathcal{S}_\pi(U) = \{s \in \mathrm{Hom}_{\mathsf{Top}}(U, E) : \pi(s(x)) = x \text{ for all } x \in U\}$$

and on morphisms by

$$\begin{aligned} \mathcal{S}_\pi(U \subseteq V) : \mathcal{S}_\pi(V) &\to \mathcal{S}_\pi(U) \\ s &\mapsto s|_U. \end{aligned}$$

This is called the *sheaf of sections* of π.

One of the central theorems of sheaf theory is that every sheaf on a space arises as the sheaf of sections of some continuous function. We will prove this later – for now, we note that this informs some common terminology.

Notation. When $\mathcal{F}$ is a (pre)sheaf on X and U is an open subset of X, we call the elements of $\mathcal{F}(U)$ *local sections* of $\mathcal{F}$ over U. When $U = X$, we call the elements of $\mathcal{F}(X)$ *global sections* of $\mathcal{F}$. When $U \subseteq V$ is an inclusion of open sets, the map

$$\mathcal{F}(U \subseteq V) : \mathcal{F}(V) \to \mathcal{F}(U)$$

is called *restriction*, and may be denoted res^V_U. We may even write $s|_U$ for $\mathrm{res}^V_U(s)$ when $s \in \mathcal{F}(V)$, even when s is not literally a function!

Approx. Start of Lecture 3, Sep 1, 2026

1.2.1 Operations and Functors on (Pre-)Sheaves

Whenever $\mathcal{D}$ is (co)complete, i.e. admits all small (co)limits, the category $\mathsf{PSh}(\mathcal{C}, \mathcal{D})$ is also (co)complete, and these (co)limits are computed pointwise. In particular, for a diagram $F : \mathcal{I} \to \mathsf{PSh}(\mathcal{C}, \mathcal{D})$ (with $\mathcal{I}$ a small category), we have

$$(\lim F)(c) \cong \lim_{i \in \mathcal{I}} F(i)(c) \qquad (\operatorname{colim} F)(c) \cong \operatorname*{colim}_{i \in \mathcal{I}} F(i)(c)$$

for all objects $c \in \mathcal{C}$. In particular, we may form products, coproducts, pullbacks, equalizers, etc. of presheaves quite easily. The same is true for *limits* of sheaves on spaces[4].

Proposition 1.2.10. *Let X be a space and let $\mathcal{D}$ be an algebraic category. Then $\mathsf{Sh}(X, \mathcal{D})$ is complete, and limits in $\mathsf{Sh}(X, \mathcal{D})$ are computed pointwise.*

Proof. Exercise. □

This tells us, for example, that morphisms of sheaves of abelian groups have kernels, which are computed pointwise, i.e. $\ker(\varphi)(U) = \ker(\varphi_U : \mathcal{F}(U) \to \mathcal{F}'(U))$ for any morphism $\varphi : \mathcal{F} \to \mathcal{F}'$ of sheaves of abelian groups and any open set $U \subseteq X$.

There are also constructions that allow us to move between different categories of (pre)sheaves.

Proposition 1.2.11. *Let $F : \mathcal{C} \to \mathcal{C}'$ be any functor, and let $\mathcal{D}$ be any category. There is a functor $F^* : \mathsf{PSh}(\mathcal{C}', \mathcal{D}) \to \mathsf{PSh}(\mathcal{C}, \mathcal{D})$ given by precomposition with F^{op}.*

Now assume moreover that $\mathcal{C}$ and $\mathcal{C}'$ are small.

(i) If $\mathcal{D}$ is cocomplete, then F^* admits a left adjoint $F_! : \mathsf{PSh}(\mathcal{C}, \mathcal{D}) \to \mathsf{PSh}(\mathcal{C}', \mathcal{D})$ defined by

$$(F_!\mathcal{F})(c') = \operatorname*{colim}_{(c,u)\in(c'\downarrow F)^{\mathrm{op}}} \mathcal{F}(c).$$

(ii) If $\mathcal{D}$ is complete, then F^* admits a right adjoint $F_* : \mathsf{PSh}(\mathcal{C}, \mathcal{D}) \to \mathsf{PSh}(\mathcal{C}', \mathcal{D})$ defined by

$$(F_*\mathcal{F})(c') = \lim_{(c,u)\in(F\downarrow c')^{\mathrm{op}}} \mathcal{F}(c).$$

Proof. Exercise for experts. □

This proposition is stated in large generality so that we can use it in other contexts later – it is not necessary to fully understand the definitions of $F_!$ and F_*; we only need to know they exist[5]. For now, we only need the functors F^* and $F_!$ in the special case that $F = f^{-1} : \mathsf{Open}(Y) \to \mathsf{Open}(X)$, where $f : X \to Y$ is a continuous map between spaces.

[4] Actually, $\mathsf{Sh}(X)$ is both complete and cocomplete, but the colimits are not computed pointwise. We'll get into this later.

[5] Also, F_* almost never appears in practice.

Corollary 1.2.12. *Let $f : X \to Y$ be a continuous map of spaces, and let $\mathcal{D}$ be a category which admits all filtered colimits*[6]*. Then $f^{-1} : \mathsf{Open}(Y) \to \mathsf{Open}(X)$ is a functor between small categories, and we obtain an adjunction*[7]

$$\begin{tikzcd} \mathsf{PSh}(X,\mathcal{D}) \arrow[d, shift left, "f_* := (f^{-1})^*"] \\ \mathsf{PSh}(Y,\mathcal{D}) \arrow[u, shift left, "(f^{-1})_! =: f^{-1}_{\mathrm{pre}}"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

(the left adjoint is written on the left-hand side). These functors are given on objects by

$$(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$$

and

$$(f^{-1}_{\mathrm{pre}}\mathcal{F})(U) = \operatorname*{colim}_{V \supseteq f(U)} \mathcal{F}(V).$$

Note that if $f(U)$ is open in Y, then $(f^{-1}_{\mathrm{pre}}\mathcal{F})(U) = \mathcal{F}(f(U))$. It is clear from the definitions that these constructions are (pseudo)functorial: for any composable pair of continuous maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have

$$(g \circ f)_* = g_* \circ f_*$$

and it follows that

$$(g \circ f)^{-1}_{\mathrm{pre}} \cong f^{-1}_{\mathrm{pre}} \circ g^{-1}_{\mathrm{pre}}.$$

As a special case, we may consider the inclusion of a point $x \in X$, which we will denote $i_x : \mathrm{pt} \to X$.

Definition 1.2.13. Let $\mathcal{D}$ be a category admitting all filtered colimits. For any space X and any point $x \in X$, we define the *stalk* of a presheaf $\mathcal{F} \in \mathsf{PSh}(X,\mathcal{D})$ at x to be the global sections of the presheaf $(i_x)^{-1}_{\mathrm{pre}}(\mathcal{F})$, i.e.

$$\mathcal{F}_x := (i_x)^{-1}_{\mathrm{pre}}(\mathcal{F})(\mathrm{pt}) = \operatorname*{colim}_{U \ni x} \mathcal{F}(U).$$

This yields the *stalk functor* $\mathcal{F} \mapsto \mathcal{F}_x : \mathsf{PSh}(X,\mathcal{D}) \to \mathcal{D}$.

Stalks are very important invariants of a (pre)sheaf. In nice cases, filtered colimits in $\mathcal{D}$ commute with finite limits, so the stalk functor preserves finite limits. We record this briefly since it is often useful.

Proposition 1.2.14. *Let $\mathcal{D}$ be an algebraic category. Then $f^{-1}_{\mathrm{pre}} : \mathsf{PSh}(Y,\mathcal{D}) \to \mathsf{PSh}(X,\mathcal{D})$ preserves finite limits for any continuous map f, and in particular the stalk functor $\mathcal{F} \mapsto \mathcal{F}_x : \mathsf{PSh}(X,\mathcal{D}) \to \mathcal{D}$ preserves finite limits for any point $x \in X$.*

---

[6] For example, any algebraic category. Proposition 1.2.11 asks that $\mathcal{D}$ admit all colimits, but filtered colimits suffice for this corollary.

[7] It's worth acknowledging that this notation is very confusing.

Proof. Abstract nonsense. □

Definition 1.2.15. If $U \subseteq X$ is an open set containing a point x, then U is one of the open sets indexing the colimit defining the stalk $\mathcal{F}_x$, and so we have a canonical map

$$\mathcal{F}(U) \to \mathcal{F}_x$$

which is natural in $\mathcal{F}$.

When $\mathcal{D}$ is one of our algebraic categories[8], we write $s \mapsto s_x$ for the action of this map. We call s_x the *germ*[9] of the section s at x.

Intuitively, if we think of s as some kind of function locally defined on U, its germ s_x remembers only the behavior of s at points "arbitrarily close" to x.

Proposition 1.2.16. *Let $s, t \in \mathcal{F}(U)$ be local sections of a presheaf $\mathcal{F} \in \mathsf{PSh}(X, \mathcal{D})$, where $\mathcal{D}$ is algebraic. Let $x \in U$. We have $s_x = t_x$ in $\mathcal{F}_x$ if and only if there is some open neighborhood V of x such that $V \subseteq U$ and $s|_V = t|_V$.*

Proof. Exercise (this just boils down to how colimits are constructed in our algebraic categories). □

In fact, the above proposition can be generalized: we can take $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(U')$ for any two open sets $U, U' \subseteq X$ containing x, and find that $s_x = t_x$ if and only if there is some open neighborhood $V \subseteq U \cap U'$ of x such that $s|_V = t|_V$. In particular, note that $(s|_V)_x = s_x$ whenever $x \in V \subseteq U$ and $s \in \mathcal{F}(U)$.

We finish with an omnibus theorem collecting some important facts about sheaves on spaces.

Theorem 1.2.17. *Let $\mathcal{D}$ be an algebraic category and X, Y topological spaces. Then:*

(i) For any continuous function $f : X \to Y$, the functor $f_* : \mathsf{PSh}(X, \mathcal{D}) \to \mathsf{PSh}(Y, \mathcal{D})$ restricts to a functor $f_* : \mathsf{Sh}(X, \mathcal{D}) \to \mathsf{Sh}(Y, \mathcal{D})$.

(ii) For any point $x \in X$, the functor $(i_x)^{-1}_{\text{pre}} : \mathsf{PSh}(X, \mathcal{D}) \to \mathsf{PSh}(\text{pt}, \mathcal{D})$ restricts to a functor $(i_x)^{-1}_{\text{pre}} : \mathsf{Sh}(X, \mathcal{D}) \to \mathsf{Sh}(\text{pt}, \mathcal{D})$. Noting that taking global sections is an equivalence $\mathsf{Sh}(\text{pt}, \mathcal{D}) \to \mathcal{D}$, we can view $(i_x)^{-1}_{\text{pre}}$ as the stalk functor $\mathcal{F} \mapsto \mathcal{F}_x$.

(iii) For any open set $U \subseteq X$ and any sheaf $\mathcal{F} \in \mathsf{Sh}(X, \mathcal{D})$, the morphism $\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x$ given by taking germs is injective.

[8] Note that all of our algebraic categories are concrete, i.e. objects in $\mathcal{D}$ have underlying sets and therefore elements.

[9] All of the terminology here is agricultural. A sheaf is a collected bundle of stalks of (e.g.) wheat, and each stalk of wheat has many germs (basically, the seeds/grains at the end of the stalk).

(iv) *Let $\varphi : \mathcal{F} \to \mathcal{F}'$ be a morphism in $\mathsf{Sh}(X, \mathcal{D})$. Then φ is an isomorphism if and only if the induced morphism on stalks $\varphi_x : \mathcal{F}_x \to \mathcal{F}'_x$ is an isomorphism for all $x \in X$.*

(v) *Let $\varphi_1, \varphi_2 : \mathcal{F} \to \mathcal{F}'$ be morphisms in $\mathsf{Sh}(X, \mathcal{D})$. We have $\varphi_1 = \varphi_2$ if and only if $(\varphi_1)_x = (\varphi_2)_x$ for all $x \in X$.*

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Proof. (i) Exercise.

(ii) Exercise.

(iii) Let $s, t \in \mathcal{F}(U)$ be such that $s_x = t_x$ for all $x \in U$. Let $\mathcal{V}$ be the collection of all open subsets $V \subseteq U$ such that $s|_V = t|_V$. Since $s_x = t_x$ for all $x \in U$, Proposition 1.2.16 tells us that $\mathcal{V}$ is an open cover of U. Now we apply the sheaf condition: s and t are local sections of $\mathcal{F}$ on U which agree on all pieces of the open cover $\mathcal{V}$, so $s = t$.

(iv) If φ is an isomorphism, then φ_x is an isomorphism for all $x \in X$ (because the stalk functors are functors). Conversely, suppose that φ_x is an isomorphism for all $x \in X$. Since $\mathcal{D}$ is algebraic, it suffices to show that the underlying morphism of presheaves of sets is an isomorphism, i.e. that $\varphi_U : \mathcal{F}(U) \to \mathcal{F}'(U)$ is a bijection for all U. So, let $U \in \mathsf{Open}(X)$ be arbitrary. By naturality of taking germs and (iii), we have a commutative diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi_U"] \arrow[d, hook] & \mathcal{F}'(U) \arrow[d, hook] \\ \prod_{x \in U} \mathcal{F}_x \arrow[r, "\cong", "\prod_{x\in U} \varphi_x"'] & \prod_{x \in U} \mathcal{F}'_x \end{tikzcd}$$

showing that φ_U is injective. To show that φ_U is surjective, let $s' \in \mathcal{F}'(U)$. For each $x \in U$, we have $s'_x \in \mathcal{F}'_x$, and since φ_x is an isomorphism, there exists a unique $a_x \in \mathcal{F}_x$ such that $\varphi_x(a_x) = s'_x$. By the definition of stalks, for each x there exists an open neighborhood V_x of x and a local section $\sigma_x \in \mathcal{F}(V_x)$ such that $(\sigma_x)_x = a_x$. Without loss of generality, we may assume $V_x \subseteq U$ (else replace V_x with $V_x \cap U$ and σ_x with $\sigma_x|_{V_x \cap U}$). By naturality of taking germs, we have

$$(\varphi_{V_x}(\sigma_x))_x = \varphi_x((\sigma_x)_x) = \varphi_x(a_x) = s'_x.$$

Now by Proposition 1.2.16, for each $x \in U$ there exists an open neighborhood $W_x \subseteq V_x$ of x such that

$$\varphi_{W_x}(\sigma_x|_{W_x}) = \varphi_{V_x}(\sigma_x)|_{W_x} = s'|_{W_x}.$$

Then $\{W_x\}_{x\in U}$ is an open cover of U. Now let $x_1, x_2 \in U$ be arbitrary. We claim that

$$\sigma_{x_1}|_{W_{x_1}\cap W_{x_2}} = \sigma_{x_2}|_{W_{x_1}\cap W_{x_2}},$$

which will allow us to glue the local sections $\{\sigma_x|_{W_x}\}_{x\in U}$ together.

Let $z \in W_{x_1} \cap W_{x_2}$ be arbitrary. By naturality of taking germs and Proposition 1.2.16, we have

$$\begin{aligned}
\varphi_z((\sigma_{x_1}|_{W_{x_1}\cap W_{x_2}})_z) &= (\varphi_{W_{x_1}\cap W_{x_2}}(\sigma_{x_1}|_{W_{x_1}\cap W_{x_2}}))_z \\
&= (\varphi_{W_{x_1}}(\sigma_{x_1}|_{W_{x_1}})|_{W_{x_1}\cap W_{x_2}})_z \\
&= (s'|_{W_{x_1}}|_{W_{x_1}\cap W_{x_2}})_z \\
&= s'_z
\end{aligned}$$

and symmetrically $\varphi_z((\sigma_{x_2}|_{W_{x_1}\cap W_{x_2}})_z) = s'_z$. Since $\varphi_z : \mathcal{F}_z \to \mathcal{F}'_z$ is an isomorphism, we conclude that

$$(\sigma_{x_1}|_{W_{x_1}\cap W_{x_2}})_z = (\sigma_{x_2}|_{W_{x_1}\cap W_{x_2}})_z.$$

By (iii), we conclude that $\sigma_{x_1}|_{W_{x_1}\cap W_{x_2}} = \sigma_{x_2}|_{W_{x_1}\cap W_{x_2}}$, as desired.

Thus, we may apply the sheaf condition to obtain $s \in \mathcal{F}(U)$ such that $s|_{W_x} = \sigma_x|_{W_x}$ for all $x \in U$. Note that

$$\varphi_U(s)|_{W_x} = \varphi_{W_x}(s|_{W_x}) = \varphi_{W_x}(\sigma_x|_{W_x}) = s'|_{W_x}$$

for all $x \in U$. Since $\mathcal{F}'$ is a sheaf and $\{W_x\}_{x\in U}$ is an open cover of U, we conclude that $\varphi_U(s) = s'$.

(v) Let $\psi : \mathcal{E} \to \mathcal{F}$ be the equalizer of φ_1 and φ_2 in $\mathsf{Sh}(X, \mathcal{D})$. By Proposition 1.2.14, we have that ψ_x is the equalizer of $(\varphi_1)_x$ and $(\varphi_2)_x$ in $\mathcal{D}$ for all $x \in X$. Now ψ is an isomorphism if and only if $\varphi_1 = \varphi_2$, and ψ_x is an isomorphism if and only if $(\varphi_1)_x = (\varphi_2)_x$. The result follows from (iv). □

Corollary 1.2.18. *A sheaf $\mathcal{F}$ is terminal (in the category $\mathsf{Sh}(X, \mathcal{D})$) if and only if $\mathcal{F}_x$ is terminal in $\mathcal{D}$ for all $x \in X$.*

Proof. First, note that the terminal presheaf $*$ (the constant functor whose value is the terminal object in $\mathcal{D}$) is also the terminal object of $\mathsf{Sh}(X, \mathcal{D})$, by Proposition 1.2.10. We may compute directly that $*_x$ is terminal in $\mathcal{D}$ for all $x \in X$. Then let $\mathcal{F} \in \mathsf{Sh}(X, \mathcal{D})$ be arbitrary. There is a unique morphism of sheaves $\mathcal{F} \to *$, which is an isomorphism if and only if $\mathcal{F}$ is terminal. By Theorem 1.2.17, we conclude that $\mathcal{F}$ is terminal if and only if $\mathcal{F}_x \to *_x$ is an isomorphism for all $x \in X$. Since $*_x$ is terminal in $\mathcal{D}$, this is equivalent to $\mathcal{F}_x$ being terminal in $\mathcal{D}$ for all $x \in X$. □

Slogan 2. Sheaves are governed by their stalks.

Remark 1.2.19. This is a good time to return to our definition of "algebraic category". One may wonder why we don't simply define $\mathsf{Sh}(X, \mathcal{D})$ for $\mathcal{D}$ an arbitrary category – our Definition 1.2.8 makes perfect sense in general[10]. There are two reasons we don't do this.

- In order to define stalks, we need $\mathcal{D}$ to admit all filtered colimits.
- Even if $\mathcal{D}$ is complete and cocomplete, so that we may define stalks and all of our nice constructions, Theorem 1.2.17(iv) (and thus Slogan 2) may fail. See Exercise 1.2.19 for an example.

Really, the only category of sheaves on X that matters is $\mathsf{Sh}(X, \mathsf{Set})$ – sheaves with values in other categories are correctly interpreted "internally", i.e. as sheaves of sets with extra structure (see Exercise 1.2.8). For our algebraic categories, it turns out that it is equivalent to take the "external" definition of sheaves with values in $\mathcal{D}$ or the "internal" definition of sheaves of sets with extra structure (and often convenient to define things in the external way).

1.2.2 Open Embeddings

We have seen (in Theorem 1.2.17) that for any continuous map $f : X \to Y$, the functor $f_* : \mathsf{PSh}(X, \mathcal{D}) \to \mathsf{PSh}(Y, \mathcal{D})$ preserves sheaves. Its left adjoint f^{-1}_{pre}, however, does not preserve sheaves in general (see Exercise 1.2.17). If f is an open embedding, this problem goes away.

Proposition 1.2.20. *Let $i : U \to X$ be an open embedding[11]. Then the functor $i^{-1}_{\mathrm{pre}} : \mathsf{PSh}(X, \mathcal{D}) \to \mathsf{PSh}(U, \mathcal{D})$ preserves sheaves, and thus restricts to a functor $i^{-1}_{\mathrm{pre}} : \mathsf{Sh}(X, \mathcal{D}) \to \mathsf{Sh}(U, \mathcal{D})$ which is left adjoint to $i_* : \mathsf{Sh}(U, \mathcal{D}) \to \mathsf{Sh}(X, \mathcal{D})$.*

Proof. We may compute directly that

$$(i^{-1}_{\mathrm{pre}}\mathcal{F})(V) = \operatorname*{colim}_{W \supseteq i(V)} \mathcal{F}(W) = \mathcal{F}(i(V))$$

for any open set $V \subseteq U$. Then the sheaf condition for $\mathcal{F}$ on X implies the sheaf condition for $i^{-1}_{\mathrm{pre}}\mathcal{F}$ on U. The adjunction follows from the fact that $\mathsf{Sh}(X, \mathcal{D})$ is a full subcategory of $\mathsf{PSh}(X, \mathcal{D})$. □

[10] Even if $\mathcal{D}$ does not admit all equalizers, we can simply demand that each required equalizer does exist.

[11] In other words, up to homeomorphism, i is the inclusion of an open subspace of X.

In the case that $i : U \to X$ is literally the inclusion of an open subspace (as opposed to an arbitrary open embedding), we often write $\mathcal{F} \mapsto \mathcal{F}|_U$ for the action of the functor i^{-1}_{pre}, and we call $\mathcal{F}|_U$ the *restriction* of $\mathcal{F}$ to U.

The fact that $i^{-1}_{\text{pre}} : \mathsf{Sh}(X, \mathcal{D}) \to \mathsf{Sh}(U, \mathcal{D})$ is well-defined when i is an open embedding is explained by an underlying relationship between $\mathsf{Open}(U)$ and $\mathsf{Open}(X)$, which we will now explore.

Proposition 1.2.21. *Let $i : U \to X$ be an open embedding. Then the functor $i^{-1} : \mathsf{Open}(X) \to \mathsf{Open}(U)$ is right adjoint to the functor $i : \mathsf{Open}(U) \to \mathsf{Open}(X)$.*

Proof. Elementary. □

Remark 1.2.22. In fact, for any open map $f : X \to Y$, the functor $f^{-1} : \mathsf{Open}(Y) \to \mathsf{Open}(X)$ is right adjoint to the functor $f : \mathsf{Open}(X) \to \mathsf{Open}(Y)$. So, the work in this subsection can be generalized to this more general situation.

Proposition 1.2.23. *Let $\mathcal{C}$ and $\mathcal{C}'$ be small categories, and let $\mathcal{D}$ be any category. Let*

$$\begin{tikzcd} \mathcal{C} \arrow[d, shift right, "F"'] \arrow[d, phantom, "\dashv"] \\ \mathcal{C}' \arrow[u, shift right, "G"'] \end{tikzcd}$$

be an adjunction (F is the left adjoint). Then the functor $F^* : \mathsf{PSh}(\mathcal{C}', \mathcal{D}) \to \mathsf{PSh}(\mathcal{C}, \mathcal{D})$ is left adjoint to the functor $G^* : \mathsf{PSh}(\mathcal{C}, \mathcal{D}) \to \mathsf{PSh}(\mathcal{C}', \mathcal{D})$.

Proof. The adjunction $F \dashv G$ is determined by a unit $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ and a counit $\varepsilon : F \circ G \to \mathrm{id}_{\mathcal{C}'}$. The adjunction $F^* \dashv G^*$ has unit

$$\mathrm{id}_{\mathsf{PSh}(\mathcal{C}', \mathcal{D})} = \mathrm{id}^*_{\mathcal{C}'} \to (F \circ G)^* = G^* \circ F^*$$

given by precomposition with ε, and counit

$$F^* \circ G^* = (G \circ F)^* \to \mathrm{id}^*_{\mathcal{C}} = \mathrm{id}_{\mathsf{PSh}(\mathcal{C}, \mathcal{D})}$$

given by precomposition with η. We leave the verification of the triangle identities as an exercise. □

I apologize for the confusing overloading of notation: here i denotes both the map of spaces $U \to X$ as well as the map of posets $\mathsf{Open}(U) \to \mathsf{Open}(X)$. In the notation of Proposition 1.2.11, we have (by definition)

$$i^{-1}_{\text{pre}} : \mathsf{PSh}(X, \mathcal{D}) \to \mathsf{PSh}(U, \mathcal{D}) \quad = \quad (i^{-1} : \mathsf{Open}(X) \to \mathsf{Open}(U))_!$$

and

$$i_* : \mathsf{PSh}(U, \mathcal{D}) \to \mathsf{PSh}(X, \mathcal{D}) \quad = \quad (i^{-1} : \mathsf{Open}(X) \to \mathsf{Open}(U))^*.$$

Now the adjunction $i \dashv i^{-1}$ of Proposition 1.2.21 gives us an adjunction $i^* \dashv (i^{-1})^*$ by Proposition 1.2.23, while we also have the adjunction $(i^{-1})_! \dashv (i^{-1})^*$ from Proposition 1.2.11. By uniqueness of adjoints, we conclude that $(i^{-1})_! \cong i^*$. This makes it almost immediate that i^{-1}_{pre} preserves sheaves, because it is given by i^* and $i : \mathsf{Open}(U) \to \mathsf{Open}(X)$ sends open covers to open covers[12].

We get even more: i^* admits a left adjoint $i_!$, extending our adjoint pair $i^{-1}_{\text{pre}} \dashv i_*$ to an adjoint triple

$$\underbrace{i_!}_{i_!} \dashv \underbrace{i^* = (i^{-1})_!}_{i^{-1}_{\text{pre}}} \dashv \underbrace{i_* = (i^{-1})^*}_{i_*}$$

of functors between categories of presheaves. The underbraces indicate the standard notation for each functor, grouping together various functors written in the notation of Proposition 1.2.11.

Proposition 1.2.24. *Let $\mathcal{D}$ be a cocomplete category. The functor*

$$i_! : \mathsf{PSh}(U, \mathcal{D}) \to \mathsf{PSh}(X, \mathcal{D})$$

is given on objects by

$$(i_!\mathcal{F})(V) = \begin{cases} \mathcal{F}(V) & : V \subseteq U \\ 0 & : V \not\subseteq U \end{cases}$$

where 0 denotes the initial object of $\mathcal{D}$. The operation $i_!$ is called extension by zero. *If $\mathcal{D} = \mathsf{Set}$, then $i_!$ preserves sheaves.*

Proof. From Corollary 1.2.12, we have

$$(i_!\mathcal{F})(V) = \operatorname*{colim}_{i(W) \supseteq V} \mathcal{F}(W) = \begin{cases} \mathcal{F}(V) & : V \subseteq U \\ 0 & : V \not\subseteq U \end{cases}.$$

If $\mathcal{D} = \mathsf{Set}$ then $0 = \varnothing$, and the sheaf condition for $i_!\mathcal{F}$ follows from the sheaf condition for $\mathcal{F}$ together with the fact that if $V \not\subseteq U$, then any open cover of V must contain some open set $W \not\subseteq U$. ∎

The restriction functor $\mathcal{F} \mapsto \mathcal{F}|_U$ also allows us to define an enrichment of the category of (pre)sheaves.

Proposition 1.2.25. *Let X be a space and $\mathcal{D}$ a category enriched over a base $\mathcal{V}$ which is complete, cocomplete, and closed symmetric monoidal.*

First, the category $\mathsf{PSh}(X, \mathcal{D})$ is naturally enriched over $\mathcal{V}$, with

$$\mathrm{Hom}(\mathcal{F}, \mathcal{F}') := \int_{U \in \mathsf{Open}(X)^{\mathrm{op}}} \mathrm{Hom}(\mathcal{F}(U), \mathcal{F}'(U)).$$

[12] This is the same reason that $f_* = (f^{-1})^*$ preserves sheaves for any continuous map f.

Moreover, $\mathsf{PSh}(X, \mathcal{D})$ *is naturally enriched over* $\mathsf{PSh}(X, \mathcal{V})$*, with*

$$\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{F}')(U) := \mathrm{Hom}_{\mathsf{PSh}(U,\mathcal{D})}(\mathcal{F}|_U, \mathcal{F}'|_U)$$

for each open set $U \subseteq X$*. Note that the global sections of* $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{F}')$ *are precisely the enriched hom* $\mathrm{Hom}(\mathcal{F}, \mathcal{F}') \in \mathcal{V}$*.*

If $\mathcal{D}$ *and* $\mathcal{V}$ *are moreover algebraic*[13]*, then the same holds for sheaves:* $\mathsf{Sh}(X, \mathcal{D})$ *is enriched over* $\mathcal{V}$ *and over* $\mathsf{Sh}(X, \mathcal{V})$*.*

Proof. It is clear that $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{F}')$ is a well-defined presheaf on X when $\mathcal{F}$ and $\mathcal{F}'$ are presheaves[a]. We leave it as an exercise to show that $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{F}')$ is a sheaf whenever $\mathcal{F}'$ is a sheaf. □

[a] Make sure you know what the restriction maps are!

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Be not afraid of the first formula above. In the case $\mathcal{V} = \mathsf{Set}$, the enrichment over $\mathcal{V}$ is just the usual category structure[14], by the end-formula for natural transformations. The other important case is $\mathcal{V} = \mathsf{Ab}$, which gives the category of (pre)sheaves of abelian groups on X a natural enrichment over itself – all this is doing is noting that the set of morphisms between two presheaves of abelian groups is itself an abelian group!

It is also worth noting that we get enrichments of automorphism groups of (pre)sheaves:

$$\underline{\mathrm{Aut}}(\mathcal{F})(U) := \mathrm{Aut}_{\mathsf{PSh}(U,\mathcal{D})}(\mathcal{F}|_U)$$

defines a presheaf of groups[15] on X when $\mathcal{F}$ is a presheaf, and a sheaf of groups on X when $\mathcal{F}$ is a sheaf[16].

1.2.3 Gluing

Proposition 1.2.26. *Let* X *be a space and let* $\mathcal{D}$ *be an algebraic category. Let* $\mathcal{B}$ *be a basis*[17] *for the topology on* X*. We may view* $\mathcal{B}$ *as a full subcategory of* $\mathsf{Open}(X)$*. Then the restriction functor* $\mathsf{Sh}(X, \mathcal{D}) \to \mathsf{PSh}(\mathcal{B}, \mathcal{D})$ *is fully faithful.*

Proof. Exercise. □

We note that this does not hold for presheaves on X! In part, this proposition says that a sheaf is determined by its values on a basis of the topology – this

[13] The only options for $\mathcal{V}$ here are Set and Ab.
[14] A category is the same thing as a Set-enriched category.
[15] Here we must take $\mathcal{D}$ as an ordinary category, ignoring any possible enrichment.
[16] Exercise!
[17] Recall that a basis $\mathcal{B}$ for the topology on X is a collection of open subsets of X such that every open subset U of X can be written as a union $\bigcup \mathcal{V}$ for some $\mathcal{V} \subseteq \mathcal{B}$.

requires the sheaf condition. It's also important to note that this restriction functor is not essentially surjective, i.e. not every presheaf on a basis of the topology extends to a sheaf on the whole space.

We can characterize exactly which presheaves on $\mathcal{B}$ come from sheaves on X. It is *almost* enough to simply say that they are the presheaves on $\mathcal{B}$ which "satisfy the sheaf condition" – the problem is that $\mathcal{B}$ need not be closed under intersection, so terms like $\mathcal{F}(B_1 \cap B_2)$ naturally appearing in a naive version of the sheaf condition are not well-defined. However, since $\mathcal{B}$ is a basis, any intersection $B_1 \cap B_2$ can itself be covered by elements of $\mathcal{B}$!

Definition 1.2.27. Let X be a space, let $\mathcal{B}$ be a basis for the topology on X, and let $\mathcal{D}$ be an algebraic category. A presheaf $\mathcal{F} \in \mathsf{PSh}(\mathcal{B}, \mathcal{D})$ is a *sheaf on* $\mathcal{B}$ if the following sheaf condition holds:

($\star$) For every $B \in \mathcal{B}$, every open cover $B = \bigcup_\alpha B_\alpha$ with each $B_\alpha \in \mathcal{B}$, and every collection of open covers[18]

$$B_\alpha \cap B_\beta = \bigcup_\gamma B_{\alpha,\beta,\gamma}$$

with each $B_{\alpha,\beta,\gamma} \in \mathcal{B}$, the diagram

$$\mathcal{F}(B) \to \prod_\alpha \mathcal{F}(B_\alpha) \rightrightarrows \prod_{\alpha,\beta} \prod_\gamma \mathcal{F}(B_{\alpha,\beta,\gamma})$$

is an equalizer diagram in $\mathcal{D}$.

We write $\mathsf{Sh}(\mathcal{B}, \mathcal{D})$ for the full subcategory of $\mathsf{PSh}(\mathcal{B}, \mathcal{D})$ spanned by the sheaves.

Remark 1.2.28. Suppose moreover that $\mathcal{B}$ is closed under finite intersections. Then $\mathcal{F} \in \mathsf{PSh}(\mathcal{B}, \mathcal{D})$ is a sheaf if and only if, for every $B \in \mathcal{B}$ and every cover $B = \bigcup_\alpha B_\alpha$ by elements of $\mathcal{B}$, the diagram

$$\mathcal{F}(B) \to \prod_\alpha \mathcal{F}(B_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathcal{F}(B_\alpha \cap B_\beta)$$

is an equalizer diagram in $\mathcal{D}$.

Theorem 1.2.29. *Restriction along the inclusion $\mathcal{B} \subseteq \mathsf{Open}(X)$ is an equivalence of categories*

$$\mathsf{Sh}(X, \mathcal{D}) \xrightarrow{\simeq} \mathsf{Sh}(\mathcal{B}, \mathcal{D}).$$

Proof. The inverse is given by right Kan extension along the inclusion $\mathcal{B} \to \mathsf{Open}(X)$: for $\mathcal{F} \in \mathsf{Sh}(\mathcal{B}, \mathcal{D})$ and $U \in \mathsf{Open}(X)$, set

$$\widetilde{\mathcal{F}}(U) := \lim_{B \subseteq U} \mathcal{F}(B).$$

Then[a] $\widetilde{\mathcal{F}}$ is a sheaf on X, and the restriction of $\widetilde{\mathcal{F}}$ to $\mathcal{B}$ is (naturally

[18] That is, we choose an open cover of $B_\alpha \cap B_\beta$. The indexing set of γ therefore depends on α and β.

isomorphic to) $\mathcal{F}$. □

[a] Exercise!

The intuitive idea in the proof above is that *any* open subset U of X may be covered by elements B_α of $\mathcal{B}$, and the intersections $B_\alpha \cap B_\beta$ may again be covered by elements of $\mathcal{B}$, allowing us to define $\widetilde{\mathcal{F}}(U)$ as the equalizer of

$$\prod_\alpha \mathcal{F}(B_\alpha) \rightrightarrows \prod_{\alpha,\beta} \prod_\gamma \mathcal{F}(B_{\alpha,\beta,\gamma}).$$

The only issue with this construction is that then the object $\widetilde{\mathcal{F}}(U)$ is only defined up to isomorphism, since each U may admit many distinct covers $\{B_\alpha\}_\alpha$, and each intersection $B_\alpha \cap B_\beta$ may admit many distinct covers $\{B_{\alpha,\beta,\gamma}\}_\gamma$. The easy fix is to choose a specific canonical cover of U by elements of $\mathcal{B}$, namely $U = \bigcup\{B \in \mathcal{B} : B \subseteq U\}$, and do likewise for the intersections. This is exactly what the Kan extension does for us.

This theorem is a very useful tool for constructing sheaves in practice. For example, this is by far the easiest way to construct the structure sheaf of an affine scheme[19]. The structure sheaf on an arbitrary scheme is then "glued" together from the structure sheaves of its affine open subschemes, so it is also worth discussing how we may glue sheaves together.

Theorem 1.2.30. *Let X be a space, let $\mathcal{D}$ be an algebraic category, and let $\{U_\alpha\}_\alpha$ be an open cover of X. Write $U_{\alpha,\beta} := U_\alpha \cap U_\beta$ and $U_{\alpha,\beta,\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$. Suppose we are given:*

(i) For each α, a sheaf $\mathcal{F}_\alpha \in \mathsf{Sh}(U_\alpha, \mathcal{D})$, and

(ii) For each pair α, β, an isomorphism

$$\varphi_{\alpha,\beta} : \mathcal{F}_\beta|_{U_{\alpha,\beta}} \xrightarrow{\cong} \mathcal{F}_\alpha|_{U_{\alpha,\beta}}$$

in $\mathsf{Sh}(U_{\alpha,\beta}, \mathcal{D})$,

satisfying the cocycle condition*: for all α, β, γ,*

$$\varphi_{\alpha,\beta}|_{U_{\alpha,\beta,\gamma}} \circ \varphi_{\beta,\gamma}|_{U_{\alpha,\beta,\gamma}} = \varphi_{\alpha,\gamma}|_{U_{\alpha,\beta,\gamma}}$$

as morphisms $\mathcal{F}_\gamma|_{U_{\alpha,\beta,\gamma}} \to \mathcal{F}_\alpha|_{U_{\alpha,\beta,\gamma}}$.

Then there is a sheaf $\mathcal{F} \in \mathsf{Sh}(X, \mathcal{D})$, together with isomorphisms

$$\psi_\alpha : \mathcal{F}|_{U_\alpha} \xrightarrow{\cong} \mathcal{F}_\alpha$$

in $\mathsf{Sh}(U_\alpha, \mathcal{D})$, such that

$$\varphi_{\alpha,\beta} \circ \left(\psi_\beta|_{U_{\alpha,\beta}}\right) = \psi_\alpha|_{U_{\alpha,\beta}}$$

for all α, β. Moreover, the data $(\mathcal{F}, (\psi_\alpha))$ is unique up to unique isomorphism (in the appropriate sense).

[19] We'll talk about this later in the notes when we introduce schemes.

Remark 1.2.31. It follows from the cocycle condition[20] that $\varphi_{\alpha,\alpha} = \mathrm{id}_{\mathcal{F}_\alpha}$ for all α, and that $\varphi_{\beta,\alpha} = \varphi_{\alpha,\beta}^{-1}$ for all α, β.

Proof. Define $\mathcal{F}$ to be the equalizer of

$$\prod_\alpha (i_\alpha)_* \mathcal{F}_\alpha \rightrightarrows \prod_{\alpha,\beta} (i_{\alpha,\beta})_* \mathcal{F}_\alpha|_{U_{\alpha,\beta}}$$

where $i_\alpha : U_\alpha \to X$ and $i_{\alpha,\beta} : U_{\alpha,\beta} \to X$ are the inclusions, and the two maps are given on local sections by

$$\begin{aligned} f_V, g_V : \prod_\alpha \mathcal{F}_\alpha(V \cap U_\alpha) &\to \prod_{\alpha,\beta} \mathcal{F}_\alpha(V \cap U_{\alpha,\beta}) \\ f_V = (s_\alpha)_\alpha &\mapsto (s_\alpha|_{V\cap U_{\alpha,\beta}})_{\alpha,\beta}, \\ g_V = (s_\alpha)_\alpha &\mapsto ((\varphi_{\alpha,\beta})_{V\cap U_{\alpha,\beta}}(s_\beta|_{V\cap U_{\alpha,\beta}}))_{\alpha,\beta}. \end{aligned}$$

Note that (α, β)-component of f is simply the projection

$$\prod_\gamma (i_\gamma)_* \mathcal{F}_\gamma \to (i_\alpha)_* \mathcal{F}_\alpha$$

onto the α-component followed by $(i_\alpha)_*$ applied to the unit

$$\mathcal{F}_\alpha \to j_* j^{-1}_{\mathrm{pre}} \mathcal{F}_\alpha,$$

of the adjunction $j^{-1}_{\mathrm{pre}} \dashv j_*$, where $j : U_{\alpha,\beta} \to U_\alpha$ is the inclusion. Similarly, the (α, β)-component of g is the projection onto the β-component, followed by $(i_\beta)_*$ applied to the unit $\mathcal{F}_\beta \to k_* k^{-1}_{\mathrm{pre}} \mathcal{F}_\beta$ (where $k : U_{\alpha,\beta} \to U_\beta$ is the inclusion), followed by $(i_{\alpha,\beta})_*$ applied to $\varphi_{\alpha,\beta}$. Thus, f and g are morphisms of presheaves, and $\mathcal{F}$ is a well-defined sheaf.

Note that, when $V \subseteq U_\alpha$, any section $(s_\beta)_\beta \in \mathcal{F}(V)$ satisfies

$$s_\beta = (\varphi_{\beta,\alpha})_{V\cap U_\beta}(s_\alpha|_{V\cap U_\beta}) \qquad (\dagger)$$

for all β, by considering the (β, α)-component of the equality $f_V((s_\beta)_\beta) = g_V((s_\beta)_\beta)$. We define[a] $\psi_\alpha : \mathcal{F}|_{U_\alpha} \to \mathcal{F}_\alpha$ by $(\psi_\alpha)_V = (s_\beta)_\beta \mapsto s_\alpha$ for all $V \in \mathsf{Open}(U_\alpha)$, and note that we have just shown that ψ_α is injective.

To show that ψ_α is surjective, we let $V \subseteq U_\alpha$ be an arbitrary open set and $t \in \mathcal{F}_\alpha(V)$ an arbitrary section. Then define

$$s_\beta := (\varphi_{\beta,\alpha})_{V\cap U_\beta}(t|_{V\cap U_\beta}) \in \mathcal{F}_\beta(V \cap U_\beta)$$

[20] Exercise!

for each β. First, we claim that $(s_\beta)_\beta \in \mathcal{F}(V)$. Indeed, we have

$$\begin{aligned} f_V((s_\beta)_\beta) = (s_\beta|_{V\cap U_{\beta,\gamma}})_{\beta,\gamma} = ((\varphi_{\beta,\alpha})_{V\cap U_\beta}(t|_{V\cap U_\beta})|_{V\cap U_{\beta,\gamma}})_{\beta,\gamma} \\ = ((\varphi_{\beta,\alpha})_{V\cap U_{\beta,\gamma}}(t|_{V\cap U_{\beta,\gamma}}))_{\beta,\gamma} \end{aligned}$$

and also

$$\begin{aligned} g_V((s_\beta)_\beta) &= ((\varphi_{\beta,\gamma})_{V\cap U_{\beta,\gamma}}(s_\gamma|_{V\cap U_{\beta,\gamma}}))_{\beta,\gamma} \\ &= ((\varphi_{\beta,\gamma})_{V\cap U_{\beta,\gamma}}((\varphi_{\gamma,\alpha})_{V\cap U_\gamma}(t|_{V\cap U_\gamma})|_{V\cap U_{\beta,\gamma}}))_{\beta,\gamma} \\ &= ((\varphi_{\beta,\gamma})_{V\cap U_{\beta,\gamma}}((\varphi_{\gamma,\alpha})_{V\cap U_{\beta,\gamma}}(t|_{V\cap U_{\beta,\gamma}})))_{\beta,\gamma} \\ &= ((\varphi_{\beta,\alpha})_{V\cap U_{\beta,\gamma}}(t|_{V\cap U_{\beta,\gamma}}))_{\beta,\gamma}, \end{aligned}$$

with the last equality coming from the cocycle condition. This shows that $(s_\beta)_\beta \in \mathcal{F}(V)$, and we have

$$(\psi_\alpha)_V((s_\beta)_\beta) = s_\alpha = (\varphi_{\alpha,\alpha})_{V\cap U_\alpha}(t|_{V\cap U_\alpha}) = (\varphi_{\alpha,\alpha})_V(t) = t.$$

The last equality here comes again from the cocycle condition. Thus $(\psi_\alpha)_V((s_\beta)_\beta) = t$, and we have shown that $(\psi_\alpha)_V$ is surjective. This shows that ψ_α is an isomorphism. Next, we must show that $\varphi_{\alpha,\beta} \circ \left(\psi_\beta|_{U_{\alpha,\beta}}\right) = \psi_\alpha|_{U_{\alpha,\beta}}$. This is a consequence of (†): when $W \subseteq U_{\alpha,\beta}$ is open, any section $(s_\gamma)_\gamma \in \mathcal{F}(W)$ satisfies

$$\begin{aligned} (\psi_\alpha)_W(s_\gamma)_\gamma &= s_\alpha \\ &= (\varphi_{\alpha,\beta})_{W\cap U_\beta}(s_\beta|_{W\cap U_\alpha}) \\ &= (\varphi_{\alpha,\beta})_W(s_\beta) \\ &= (\varphi_{\alpha,\beta})_W((\psi_\beta)_W(s_\gamma)_\gamma). \end{aligned}$$

Finally, we must establish the uniqueness of $(\mathcal{F}, (\psi_\alpha)_\alpha)$, up to unique isomorphism. Suppose $(\mathcal{F}', (\psi'_\alpha)_\alpha)$ is another such datum. Then the maps $\psi'_\alpha : \mathcal{F}'|_{U_\alpha} \to \mathcal{F}_\alpha$ correspond to maps $\mathcal{F}' \to (i_\alpha)_*\mathcal{F}_\alpha$ for each α, giving us a map

$$\mathcal{F}' \to \prod_\alpha (i_\alpha)_*\mathcal{F}_\alpha.$$

The compatibility condition $\varphi_{\alpha,\beta} \circ \left(\psi'_\beta|_{U_{\alpha,\beta}}\right) = \psi'_\alpha|_{U_{\alpha,\beta}}$ says exactly that this map factors through the equalizer $\mathcal{F}$. The morphism $\mathcal{F}' \to \mathcal{F}$ becomes an isomorphism after restriction to each U_α, since

$$\mathcal{F}_\alpha \xrightarrow{(\psi'_\alpha)^{-1}} \mathcal{F}'|_{U_\alpha} \to \mathcal{F}|_{U_\alpha} \xrightarrow{\psi_\alpha} \mathcal{F}_\alpha$$

is the identity by construction. We therefore obtain a unique isomorphism $\mathcal{F}' \xrightarrow{\cong} \mathcal{F}$ compatible with the ψ_α's and ψ'_α's, as desired. □

[a] More abstractly: $\mathcal{F}$ comes equipped with maps $\mathcal{F} \to (i_\alpha)_*\mathcal{F}_\alpha$ for all α. By the adjunction $(i_\alpha)^{-1}_{\text{pre}} \dashv (i_\alpha)_*$, these correspond to maps $\mathcal{F}|_{U_\alpha} \to \mathcal{F}_\alpha$, which are the ψ_α's.

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Corollary 1.2.32. *Let X be a space, let $\mathcal{D}$ be an algebraic category, and let $\mathcal{F}, \mathcal{F}' \in \mathsf{Sh}(X, \mathcal{D})$. Let $\{U_\alpha\}_\alpha$ be an open cover of X, and suppose we are given morphisms*

$$\varphi_\alpha : \mathcal{F}|_{U_\alpha} \to \mathcal{F}'|_{U_\alpha}$$

in $\mathsf{Sh}(U_\alpha, \mathcal{D})$ for each α, such that

$$\varphi_\alpha|_{U_\alpha \cap U_\beta} = \varphi_\beta|_{U_\alpha \cap U_\beta}$$

for all α, β. Then there is a unique morphism $\varphi : \mathcal{F} \to \mathcal{F}'$ in $\mathsf{Sh}(X, \mathcal{D})$ such that $\varphi|_{U_\alpha} = \varphi_\alpha$ for all α.

Proof. Apply Theorem 1.2.30 to the sheaves $\mathcal{F}|_{U_\alpha}$ and $\mathcal{F}'|_{U_\alpha}$ to reconstruct $\mathcal{F}$ and $\mathcal{F}'$ (up to isomorphism) from their restrictions. The maps φ_α assemble into a morphism between these reconstructions. □

Remark 1.2.33. Theorem 1.2.30 and Corollary 1.2.32 say, in a sense that can be made precise, that the functor

$$\mathsf{Sh}(X, \mathcal{D}) \to \lim \left(\prod_\alpha \mathsf{Sh}(U_\alpha, \mathcal{D}) \rightrightarrows \prod_{\alpha,\beta} \mathsf{Sh}(U_{\alpha,\beta}, \mathcal{D}) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \prod_{\alpha,\beta,\gamma} \mathsf{Sh}(U_{\alpha,\beta,\gamma}, \mathcal{D}) \right)$$

is an equivalence, where the limit[21] is taken in the 2-category of categories. In other words, $U \mapsto \mathsf{Sh}(U, \mathcal{D})$ is some sort of "higher categorical" sheaf[22] on X.

Notice that triple intersections have appeared here for the first time. For a sheaf of sets $\mathcal{F} \in \mathsf{Sh}(X)$, the agreement of two sections $s \in \mathcal{F}(U), s' \in \mathcal{F}(V)$ on a double intersection $U \cap V$ is a *condition*; we either have $s|_{U\cap V} = s'|_{U \cap V}$ or not. For a "sheaf of categories" like $U \mapsto \mathsf{Sh}(U, \mathcal{D})$, the agreement of two sections $\mathcal{F} \in \mathsf{Sh}(U, \mathcal{D}), \mathcal{F}' \in \mathsf{Sh}(V, \mathcal{D})$ on a double intersection is *data* (a choice of isomorphism $\varphi : \mathcal{F}'|_{U\cap V} \to \mathcal{F}|_{U \cap V}$); if two sheaves on $U \cap V$ are isomorphic, they may be isomorphic in many different ways. We therefore need to enforce that these isomorphisms themselves agree on triple intersections (this is the cocycle condition). An agreement between two isomorphisms is just a condition, however: two isomorphisms are either equal or not. So, we can stop there, and we don't need to consider quadruple intersections. In general, you could imagine that a "sheaf of n-categories" would require agreement data specified up to $(n+1)$-fold intersections, and agreement conditions on $(n+2)$-fold intersections.

[21] Technically, a *pseudo-limit*

[22] If you want a fancy word, $U \mapsto \mathsf{Sh}(U, \mathcal{D})$ is an example of a *stack*.

1.2.4 The Espace Étalé

We have had to be careful in checking when certain functors preserve sheaves. We've seen that, for $f : X \to Y$ a continuous function, $f_* : \mathsf{PSh}(X, \mathcal{D}) \to \mathsf{PSh}(Y, \mathcal{D})$ always preserves sheaves, but $f^{-1}_{\mathrm{pre}} : \mathsf{PSh}(Y, \mathcal{D}) \to \mathsf{PSh}(X, \mathcal{D})$ does not always preserve sheaves. So, we have the following situation:

$$\begin{tikzcd}
\mathsf{Sh}(X, \mathcal{D}) \arrow[r, hook] \arrow[d, "f_*"', shift right] & \mathsf{PSh}(X, \mathcal{D}) \arrow[d, "f_*", shift left] \\
\mathsf{Sh}(Y, \mathcal{D}) \arrow[r, hook] \arrow[u, dashed, "??", shift left] & \mathsf{PSh}(Y, \mathcal{D}) \arrow[u, "f^{-1}_{\mathrm{pre}}", shift left] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

where we would like to have a left adjoint to $f_* : \mathsf{Sh}(X, \mathcal{D}) \to \mathsf{Sh}(Y, \mathcal{D})$, but we don't always know how to construct it – it is certainly not as simple as just using f^{-1}_{pre}! It may be tempting to think that $f_* : \mathsf{Sh}(X, \mathcal{D}) \to \mathsf{Sh}(Y, \mathcal{D})$ simply does not have a left adjoint unless f^{-1}_{pre} happens to preserve sheaves.

On the other hand, we know that $f_* : \mathsf{PSh}(X, \mathcal{D}) \to \mathsf{PSh}(Y, \mathcal{D})$ preserves limits (because it is a right adjoint), and we know that limits of sheaves are computed pointwise, so $f_* : \mathsf{Sh}(X, \mathcal{D}) \to \mathsf{Sh}(Y, \mathcal{D})$ also preserves limits. So, it also seems possible that $f_* : \mathsf{Sh}(X, \mathcal{D}) \to \mathsf{Sh}(Y, \mathcal{D})$ always has a left adjoint! This turns out to be the truth.

Actually, this desired left adjoint comes from the fact that the inclusion $\mathsf{Sh}(X, \mathcal{D}) \hookrightarrow \mathsf{PSh}(X, \mathcal{D})$ has a left adjoint, called *sheafification*. In this subsection, we will construct the sheafification functor – once this is in place, the left adjoint to $f_* : \mathsf{Sh}(X, \mathcal{D}) \to \mathsf{Sh}(Y, \mathcal{D})$ has a very simple formula:

1. First, view a sheaf $\mathcal{F} \in \mathsf{Sh}(Y, \mathcal{D})$ as a presheaf.
2. Then, apply f^{-1}_{pre} to get a presheaf $f^{-1}_{\mathrm{pre}}\mathcal{F} \in \mathsf{PSh}(X, \mathcal{D})$.
3. Finally, apply the sheafification functor to get a sheaf on X.

The most efficient way to construct the sheafification functor is to use the *espace étalé*[23] construction. This is a very intuitive construction, with only two downsides:

- It only makes sense to apply directly to sheaves of sets. This is not such a big deal, since (as mentioned previously) we can always view a sheaf with values in $\mathcal{D}$ (for $\mathcal{D}$ any algebraic category) as a sheaf of sets with extra structure, and it will be easy to see that the sheafification procedure preserves this extra structure.
- It does not generalize very easily to sheaves on sites (which we'll discuss later in the course), or at least the generalization to sites is not as intuitive. We'll introduce a different construction of sheafification later in the course that works for sheaves on sites.

[23] This is French terminology. *Espace* means "space" and *étalé* means something like "spread out".

So, for the rest of this subsection, we will restrict our attention to the case $\mathcal{D} = \mathsf{Set}$.

Definition 1.2.34. Let $\mathcal{F} \in \mathsf{PSh}(X)$. The *espace étalé* of $\mathcal{F}$ is the topological space $E_{\mathcal{F}}$ whose underlying set is the disjoint union of stalks

$$E_{\mathcal{F}} := \coprod_{x \in X} \mathcal{F}_x$$

and whose topology is generated by the sets

$$s[U] := \{(x, s_x) : x \in U\} \subseteq E_{\mathcal{F}}$$

for all open sets $U \subseteq X$ and all sections $s \in \mathcal{F}(U)$. Here we write elements of the disjoint union $\coprod_{x \in X} \mathcal{F}_x$ as pairs (x, g) with $x \in X$ and $g \in \mathcal{F}_x$.

The *canonical projection* $\pi : E_{\mathcal{F}} \to X$ is defined by $\pi(x, g) := x$. It is left as an exercise to check that π is continuous.

We record some useful topological properties of the espace étalé, which should help with understanding what it is like.

Proposition 1.2.35. *Let $\mathcal{F} \in \mathsf{PSh}(X)$, and let $s \in \mathcal{F}(U)$ be a local section of $\mathcal{F}$. Then the map $f : U \to E_{\mathcal{F}}$ defined by $f(x) := (x, s_x)$ is continuous.*

Proof. To see that f is continuous, it suffices to show that $f^{-1}(s'[U'])$ is open for all open sets $U' \subseteq X$ and all local sections $s' \in \mathcal{F}(U')$. By definition of f, we have

$$f^{-1}(s'[U']) = \{x \in U \cap U' : s_x = s'_x\}.$$

This is an open set by Proposition 1.2.16. ☐

In other words, every local section of the presheaf $\mathcal{F}$ gives rise to a local section of the canonical projection $\pi : E_{\mathcal{F}} \to X$. It is worth making an explicit warning that the converse is not always true! The point is that local sections of $\mathcal{F}$ will correspond exactly to local sections of π if and only if $\mathcal{F}$ is a sheaf.

Proposition 1.2.36. *The map π is a* local homeomorphism, *i.e. for every $(x, g) \in E_{\mathcal{F}}$, there is an open neighborhood $V \subseteq E_{\mathcal{F}}$ of (x, g) such that*

$$\pi|_V : V \to \pi(V)$$

is a homeomorphism and $\pi(V)$ is open in X.

Proof. Let $(x, g) \in E_{\mathcal{F}}$ be arbitrary. By definition, there is an open neighborhood $U \subseteq X$ of x and a section $s \in \mathcal{F}(U)$ such that $s_x = g$. Then $s[U]$ is an open neighborhood of (x, g) in $E_{\mathcal{F}}$, and we claim that $\pi|_{s[U]} : s[U] \to U$ is a homeomorphism. This is straightforward: it is clear that $\pi|_{s[U]}$ is bijective with inverse $y \mapsto (y, s_y)$, and this inverse is continuous by the previous proposition. □

Local homeomorphisms are a very nice class of maps, and can be viewed as a generalization of covering spaces. In particular, any covering space is a local homeomorphism.

Corollary 1.2.37. *The fibers $\pi^{-1}(x) = \mathcal{F}_x$ of the canonical projection $\pi : E_{\mathcal{F}} \to X$ are discrete subspaces of $E_{\mathcal{F}}$.*

Proof. The fiber $\pi^{-1}(x)$ can be constructed by pullback:

$$\begin{tikzcd} \pi^{-1}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E_{\mathcal{F}} \arrow[d, "\pi"] \\ \mathrm{pt} \arrow[r, "i_x"'] & X \end{tikzcd}$$

The property of being a local homeomorphism is preserved under pullback[a], so $\pi^{-1}(x) \to \mathrm{pt}$ is a local homeomorphism. Now any point $* \in \pi^{-1}(x)$ has an open neighborhood $V \subseteq \pi^{-1}(x)$ such that $\pi|_V : V \to \mathrm{pt}$ is a homeomorphism. This implies that $V = \{*\}$, so $\pi^{-1}(x)$ is discrete. □

[a] Exercise!

The espace étalé $E_{\mathcal{F}}$ gives a canonical way to build a sheaf from a presheaf: first construct the espace étalé, and then take the sheaf of sections of the canonical projection. In fact, the second step ("take the sheaf of sections") is not doing anything at all; it is really the construction of the espace étalé that is doing the work of sheafification.

Definition 1.2.38. Let $\mathsf{Top}^{\mathrm{lh}}$ denote the category of spaces with local homeomorphisms between them. For X a space, $\mathsf{Top}^{\mathrm{lh}}/X$ (the slice category of local homeomorphisms over X) is the category whose objects are local homeomorphisms $E \to X$ and whose morphisms are commutative triangles

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & E' \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

with φ a local homeomorphism.

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Proposition 1.2.39. $\mathsf{Top}^{\mathrm{lh}}/X$ *is equal to the full subcategory of* Top/X *spanned by the local homeomorphisms with codomain X. In other words, in the above commutative triangle, it is equivalent to require only that φ is continuous (it will then automatically be a local homeomorphism).*

Proof. Exercise. □

Proposition 1.2.40. *The espace étalé construction $\mathcal{F} \mapsto E_{\mathcal{F}}$ is functorial $\mathsf{PSh}(X) \to \mathsf{Top}^{\mathrm{lh}}/X$.*

Proof. Given a morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{F}'$, we obtain morphisms of stalks $\varphi_x : \mathcal{F}_x \to \mathcal{F}'_x$, which together give a function $E_\varphi : E_{\mathcal{F}} \to E_{\mathcal{F}'}$ which commutes with the canonical projections. It is left as an exercise to show that E_φ is continuous, etc. □

Proposition 1.2.41. *The sheaf of sections construction $p \mapsto \mathcal{S}_p$ is functorial $\mathsf{Top}/X \to \mathsf{Sh}(X)$.*

Proof. Given a morphism $\varphi : E \to E'$ between maps $p : E \to X, p' : E' \to X$, we may define a morphism of sheaves $\mathcal{S}_\varphi : \mathcal{S}_p \to \mathcal{S}_{p'}$ on local sections by $s \mapsto \varphi \circ s$. It is left as an exercise to show that $\mathcal{S}_\varphi$ is well-defined, etc. □

Proposition 1.2.42. *Let $p : E \to X$ be a local homeomorphism, and let $x \in X$ be arbitrary. There is a canonical bijection $(\mathcal{S}_p)_x \to p^{-1}(x)$ given by $s_x \mapsto s(x)$.*

Proof. Let $x \in X$ and $t \in p^{-1}(x)$ be arbitrary. Since p is a local homeomorphism, there is an open neighborhood $U \subseteq E$ of t such that $p|_U : U \to p(U)$ is a homeomorphism. Now $(p|_U)^{-1} : p(U) \to U$ is a local section of p whose value at x is t. Thus, the map $s_x \mapsto s(x) : (\mathcal{S}_p)_x \to p^{-1}(x)$ is surjective. For injectivity, suppose $s_x \in (\mathcal{S}_p)_x$ is such that $s(x) = t$. Now $s^{-1}(U)$ is an open neighborhood of x, and we may assume without loss of generality that s is defined on $s^{-1}(U)$. In particular, the image of s is contained in U. Now $p|_U \circ s = \mathrm{id}_{s^{-1}(U)}$, so $s = (p|_U)^{-1}|_{s^{-1}(U)}$. □

Now we are ready to define the sheafification functor.

Definition 1.2.43. Let $\mathcal{F} \in \mathsf{PSh}(X)$. The *sheafification* of $\mathcal{F}$, denoted $\mathcal{F}^{++}$, is the sheaf of sections of the canonical projection $E_{\mathcal{F}} \to X$. In other words, the sheafification functor is the composite

$$\mathsf{PSh}(X) \to \mathsf{Top}^{\mathrm{lh}}/X \to \mathsf{Sh}(X).$$

Proposition 1.2.44. *There is a natural transformation $\eta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^{++}$ given on local sections by $s \mapsto (u \mapsto (u, s_u))$. The induced map on stalks $\mathcal{F}_x \to \mathcal{F}^{++}_x$ is an isomorphism for all $x \in X$.*

Proof. $\eta_{\mathcal{F}}$ is well-defined by Proposition 1.2.35. Naturality is left as an exercise. The isomorphism on stalks is Proposition 1.2.42. □

Consequently, $\eta_{\mathcal{F}}$ is an isomorphism if and only if $\mathcal{F}$ is a sheaf.

Theorem 1.2.45. *The espace étalé construction and the sheaf of sections construction give an equivalence of categories*

$$\mathsf{Sh}(X) \simeq \mathsf{Top}^{\mathrm{lh}}/X.$$

Proof. In one direction, $\eta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^{++}$ is an isomorphism whenever $\mathcal{F}$ is a sheaf.

In the other direction, let $p : E \to X$ be a local homeomorphism. We have a well-defined function $\beta : E_{\mathcal{S}_p} \to E$ given by $(x, s_x) \mapsto s(x)$ which commutes with the projections $\pi : E_{\mathcal{S}_p} \to X$, $p : E \to X$. On fibers, this is exactly the isomorphism $\pi^{-1}(x) = (\mathcal{S}_p)_x \to p^{-1}(x)$ of Proposition 1.2.42, so β is a bijection. If we can establish that β is continuous, we will know automatically that it is a local homeomorphism by Proposition 1.2.39, from which it follows (since local homeomorphisms are open) that β is an isomorphism in $\mathsf{Top}^{\mathrm{lh}}/X$.

For this, let $V \subseteq E$ be open, and take an arbitrary element $(x, g) \in \beta^{-1}(V)$. Choose an open neighborhood U of x and a local section $s \in \mathcal{S}_p(U)$ be such that $s_x = g$. Then $W := s^{-1}(V)$ is an open neighborhood of x. Now $(s|_W)[W]$ is open in $E_{\mathcal{S}_p}$, contains

$$(x, g) = (x, s_x) = (x, (s|_W)_x),$$

and is contained in $\beta^{-1}(V)$, since for all $w \in W$, we have

$$\beta(w, (s|_W)_w) = (s|_W)(w) = s(w) \in V.$$

This shows that $\beta^{-1}(V)$ is open, so β is continuous. □

Theorem 1.2.46. *Sheafification is left adjoint to the inclusion* $\mathsf{Sh}(X) \hookrightarrow \mathsf{PSh}(X)$.

Proof. Let $\mathcal{F} \to \mathcal{G}$ be a morphism of presheaves with $\mathcal{G}$ a sheaf. We obtain a naturality square

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d, "\eta_{\mathcal{F}}"'] & \mathcal{G} \arrow[d, "\eta_{\mathcal{G}}"', "\cong"] \\ \mathcal{F}^{++} \arrow[r, "\varphi^{++}"'] & \mathcal{G}^{++} \end{tikzcd}$$

and thus a morphism $\eta_{\mathcal{G}}^{-1} \circ \varphi^{++} : \mathcal{F}^{++} \to \mathcal{G}$ of sheaves such that

$$(\eta_{\mathcal{G}}^{-1} \circ \varphi^{++}) \circ \eta_{\mathcal{F}} = \varphi.$$

In particular, we have a well-defined injection

$$\begin{aligned} \operatorname{Hom}_{\mathsf{PSh}(X)}(\mathcal{F}, \mathcal{G}) &\to \operatorname{Hom}_{\mathsf{Sh}(X)}(\mathcal{F}^{++}, \mathcal{G}) \\ \varphi &\mapsto \eta_{\mathcal{G}}^{-1} \circ \varphi^{++} \end{aligned}$$

which is natural in both $\mathcal{F}$ and $\mathcal{G}$ by naturality of η and functoriality of $(-)^{++}$.

For surjectivity, let $\psi : \mathcal{F}^{++} \to \mathcal{G}$ be a morphism of sheaves. Then $\psi \circ \eta_{\mathcal{F}} : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, and we have

$$\eta_{\mathcal{G}}^{-1} \circ (\psi \circ \eta_{\mathcal{F}})^{++} = \eta_{\mathcal{G}}^{-1} \circ \psi^{++} \circ \eta_{\mathcal{F}}^{++}.$$

It suffices to show that this composite is equal to ψ. We claim that $\eta_{\mathcal{F}}^{++} = \eta_{\mathcal{F}^{++}}$. To see this, note that both $\eta_{\mathcal{F}}^{++}$ and $\eta_{\mathcal{F}^{++}}$ are morphisms of sheaves $\mathcal{F}^{++} \to (\mathcal{F}^{++})^{++}$, and that they fit in a naturality square

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\eta_{\mathcal{F}}"] \arrow[d, "\eta_{\mathcal{F}}"'] & \mathcal{F}^{++} \arrow[d, "\eta_{\mathcal{F}^{++}}"] \\ \mathcal{F}^{++} \arrow[r, "\eta_{\mathcal{F}}^{++}"'] & (\mathcal{F}^{++})^{++} \end{tikzcd}$$

Using the fact that $\eta_{\mathcal{F}}$ is an isomorphism on stalks, we learn that $\eta_{\mathcal{F}}^{++}$ and $\eta_{\mathcal{F}^{++}}$ agree on stalks. Thus, $\eta_{\mathcal{F}}^{++} = \eta_{\mathcal{F}^{++}}$ by Theorem 1.2.17. Now the naturality square

$$\begin{tikzcd} \mathcal{F}^{++} \arrow[r, "\psi"] \arrow[d, "\eta_{\mathcal{F}^{++}}"'] & \mathcal{G} \arrow[d, "\eta_{\mathcal{G}}"] \\ (\mathcal{F}^{++})^{++} \arrow[r, "\psi^{++}"'] & \mathcal{G}^{++} \end{tikzcd}$$

shows that

$$\eta_{\mathcal{G}}^{-1} \circ \psi^{++} \circ \eta_{\mathcal{F}}^{++} = \eta_{\mathcal{G}}^{-1} \circ \psi^{++} \circ \eta_{\mathcal{F}^{++}} = \psi,$$

and we are done. □

By construction, the unit of the adjunction is the map $\eta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^{++}$ for $\mathcal{F}$ a presheaf, and the counit of the adjunction is $\eta_{\mathcal{F}}^{-1} : \mathcal{F}^{++} \to \mathcal{F}$ for $\mathcal{F}$ a sheaf.

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With the power of sheafification at our disposal, it becomes easy to define the left adjoint to $f_* : \mathsf{Sh}(X) \to \mathsf{Sh}(Y)$ for any continuous map $f : X \to Y$.

Definition 1.2.47. Let $f : X \to Y$ be a continuous map of topological spaces. The left adjoint to $f_* : \mathsf{Sh}(X) \to \mathsf{Sh}(Y)$ is denoted f^{-1}, and given by the composite

$$\mathsf{Sh}(Y) \hookrightarrow \mathsf{PSh}(Y) \xrightarrow{f_{\text{pre}}^{-1}} \mathsf{PSh}(X) \xrightarrow{(-)^{++}} \mathsf{Sh}(X).$$

We call this functor *inverse image* or *pullback* along f.

To check that f^{-1} is indeed left adjoint to f_*, we simply see that

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{Sh}(X)}(f^{-1}\mathcal{F}, \mathcal{G}) &= \mathrm{Hom}_{\mathsf{Sh}(X)}((f_{\text{pre}}^{-1}\mathcal{F})^{++}, \mathcal{G}) \\
&\cong \mathrm{Hom}_{\mathsf{PSh}(X)}(f_{\text{pre}}^{-1}\mathcal{F}, \mathcal{G}) \\
&\cong \mathrm{Hom}_{\mathsf{PSh}(Y)}(\mathcal{F}, f_*\mathcal{G}) \\
&\cong \mathrm{Hom}_{\mathsf{Sh}(Y)}(\mathcal{F}, f_*\mathcal{G})
\end{aligned}$$

naturally in $\mathcal{F} \in \mathsf{Sh}(Y)$ and $\mathcal{G} \in \mathsf{Sh}(X)$.

Proposition 1.2.48. *$f^{-1} : \mathsf{Sh}(Y) \to \mathsf{Sh}(X)$ preserves stalks, i.e. for all points $x \in X$ there is a natural isomorphism*

$$(f^{-1}\mathcal{F})_x \cong \mathcal{F}_{f(x)}.$$

Proof. The inclusion $\mathsf{Sh}(Y) \to \mathsf{PSh}(Y)$, the functor $f_{\text{pre}}^{-1} : \mathsf{PSh}(Y) \to \mathsf{PSh}(X)$, and the sheafification functor $(-)^{++} : \mathsf{PSh}(X) \to \mathsf{Sh}(X)$ all preserve stalks. □

It also becomes easy to prove important facts about the category of sheaves. Here are a few nice results that follow easily from the espace étalé construction.

Proposition 1.2.49. *$\mathsf{Sh}(X)$ is cocomplete. Moreover, the Yoneda embedding $y : \mathsf{Open}(X) \to \mathsf{Sh}(X)$ is a (not-necessarily free) cocompletion, i.e. it is fully faithful and its image generates $\mathsf{Sh}(X)$ under colimits (in a canonical way).*

Proof. For the first claim, take a diagram $D : \mathcal{I} \to \mathsf{Sh}(X)$ with $\mathcal{I}$ a small category. Form its colimit $\operatorname{colim} D$ in $\mathsf{PSh}(X)$, and then sheafify to get $(\operatorname{colim} D)^{++} \in \mathsf{Sh}(X)$. Since sheafification is a left adjoint functor, this computes $\operatorname{colim}_{i\in\mathcal{I}} D(i)^{++}$ in $\mathsf{Sh}(X)$. Since $D(i)$ is already a sheaf, we have a natural isomorphism $D(i) \cong D(i)^{++}$, and thus $(\operatorname{colim} D)^{++}$ computes the colimit of D in $\mathsf{Sh}(X)$.

For the second claim, we use the fact that the Yoneda embedding $y : \mathsf{Open}(X) \to \mathsf{PSh}(X)$ is a free cocompletion, i.e. every presheaf on X is a small colimit (in a canonical way) of presheaves of the form $y(U)$ for $U \subseteq X$ open. Since sheafification is a left adjoint functor, it preserves colimits, so every sheaf on X is a small colimit of sheaves of the form $y(U)^{++}$. But $y(U)$ is already a sheaf, so $y(U)^{++} \cong y(U)$. □

Proposition 1.2.50. *Let $\mathcal{F}$ be a sheaf on X. The slice category $\mathsf{Sh}(X)/\mathcal{F}$ is equivalent to the category of sheaves on the espace étalé $E_{\mathcal{F}}$, and in particular is complete and cocomplete.*

Proof. By Theorem 1.2.45, we have an equivalence of categories $\mathsf{Sh}(X) \simeq \mathsf{Top}^{\mathrm{lh}}/X$. Under this equivalence, the slice category $\mathsf{Sh}(X)/\mathcal{F}$ corresponds to the slice category $(\mathsf{Top}^{\mathrm{lh}}/X)/E_{\mathcal{F}}$, which is equivalent to $\mathsf{Top}^{\mathrm{lh}}/E_{\mathcal{F}}$ by inspection. Finally, by Theorem 1.2.45 again, we have an equivalence of categories $\mathsf{Top}^{\mathrm{lh}}/E_{\mathcal{F}} \simeq \mathsf{Sh}(E_{\mathcal{F}})$. □

Corollary 1.2.51. *The slice category* $\mathsf{Sh}(X)/y(U)$ *is equivalent to* $\mathsf{Sh}(U)$.

Proof. The representable sheaf $y(U)$ corresponds to the inclusion $U \hookrightarrow X$ in $\mathsf{Top}^{\mathrm{lh}}/X$. □

Proposition 1.2.52. *Let* $f : X \to Y$ *be a continuous function. The composite*

$$\mathsf{Top}^{\mathrm{lh}}/Y \simeq \mathsf{Sh}(Y) \xrightarrow{f^{-1}} \mathsf{Sh}(X) \simeq \mathsf{Top}^{\mathrm{lh}}/X$$

is given simply by pulling back morphisms along f*.*

Proof. We have already seen (cf. Exercise 1.2.27) that local homeomorphisms are stable under pullback and that (cf. Proposition 1.2.39) $\mathsf{Top}^{\mathrm{lh}}/X$ is a full subcategory of Top/X, so "pullback along f" is a well-defined functor $\mathsf{Top}^{\mathrm{lh}}/Y \to \mathsf{Top}^{\mathrm{lh}}/X$. Let this functor be denoted f^* (just for this proof). We already know that f^* agrees with f^{-1} when $f = i_y : \mathrm{pt} \to Y$ is the inclusion of a point; in particular these "fiber functors" $i_y^* : \mathsf{Top}^{\mathrm{lh}}/Y \to \mathsf{Top}^{\mathrm{lh}}/\mathrm{pt} \simeq \mathsf{Set}$ (which agree with the stalk functors $i_y^{-1} : \mathsf{Sh}(Y) \to \mathsf{Set}$) preserve colimits. We now claim that f^* preserves colimits in general.

Let $D : \mathcal{I} \to \mathsf{Top}^{\mathrm{lh}}/Y$ be a diagram. There is a natural map

$$\mathrm{colim}(f^* \circ D) \to f^*(\mathrm{colim}\, D)$$

in $\mathsf{Top}^{\mathrm{lh}}/X$. Letting $x \in X$ be arbitrary and using the fact that fiber functors preserve colimits, we see that the induced map on fibers is isomorphic to the identity

$$\begin{aligned} i_{f(x)}^*(\mathrm{colim}\, D) \cong \mathrm{colim}(i_{f(x)}^* \circ D) \cong \mathrm{colim}(i_x^* \circ f^* \circ D) \cong i_x^*(\mathrm{colim}(f^* \circ D)) \\ \to i_x^*(f^*(\mathrm{colim}\, D)) \cong (f \circ i_x)^*(\mathrm{colim}\, D) = i_{f(x)}^*(\mathrm{colim}\, D). \end{aligned}$$

Thus, the map $\mathrm{colim}(f^* \circ D) \to f^*(\mathrm{colim}\, D)$ is an isomorphism.

We have now shown that f^* preserves colimits. By Proposition 1.2.49, in order to show that f^* and f^{-1} agree, it now suffices to show that they agree (via a natural isomorphism) on the sheaves $y(U)$ for $U \subseteq Y$ open. For this, we note that $y(U) \in \mathsf{Sh}(Y)$ corresponds to the inclusion $U \hookrightarrow Y$ in $\mathsf{Top}^{\mathrm{lh}}/Y$. Pulling back this inclusion along f gives the inclusion $f^{-1}(U) \hookrightarrow X$, which corresponds to the sheaf $y(f^{-1}(U)) \in \mathsf{Sh}(X)$. On the other hand,

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Sh}(X)}(y(f^{-1}(U)), \mathcal{G}) &\cong \mathcal{G}(f^{-1}(U)) \\ &= (f_*\mathcal{G})(U) \\ &\cong \mathrm{Hom}_{\mathsf{Sh}(Y)}(y(U), f_*\mathcal{G}) \\ &\cong \mathrm{Hom}_{\mathsf{Sh}(X)}(f^{-1}y(U), \mathcal{G})\end{aligned}$$

establishes that $y(f^{-1}(U)) \cong f^{-1}y(U)$ naturally in $U \in \mathsf{Open}(Y)$. □

So, the preimage functor f^{-1} is very easy to understand from the perspective of the espace étalé construction: it is just given by pullback! On the other hand, the direct image functor f_* is kind of tricky to define from the perspective of the espace étalé. The good news is f_* is easy to understand at the level of presheaves. Since we have both perspectives at our disposal, this is a good situation to be in.

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We can also generalize Proposition 1.2.24 to arbitrary local homeomorphisms (instead of just open embeddings).

Proposition 1.2.53. *Let $f : X \to Y$ be a local homeomorphism. Then there is an adjoint triple*

$$f_! \dashv f^{-1} \dashv f_*$$

between categories of sheaves, where $f_! : \mathsf{Sh}(X) \simeq \mathsf{Top}^{\mathrm{lh}}/X \to \mathsf{Top}^{\mathrm{lh}}/Y \simeq \mathsf{Sh}(Y)$ is the functor given by postcomposition with f.

Proof. This is a general fact: postcomposition is always left adjoint to pullback (as a functor between slice categories), whenever pullback is well-defined. □

As mentioned at the start of this subsection, much of this discussion generalizes to presheaves with values in any algebraic category $\mathcal{D}$. In particular, there is a sheafification functor $(-)^{++} : \mathsf{PSh}(X;\mathcal{D}) \to \mathsf{Sh}(X;\mathcal{D})$ which is left adjoint to the inclusion $\mathsf{Sh}(X;\mathcal{D}) \hookrightarrow \mathsf{PSh}(X;\mathcal{D})$, and an inverse image functor $f^{-1} : \mathsf{Sh}(Y;\mathcal{D}) \to \mathsf{Sh}(X;\mathcal{D})$ which is left adjoint to the direct image functor $f_* : \mathsf{Sh}(X;\mathcal{D}) \to \mathsf{Sh}(Y;\mathcal{D})$. In Exercise 1.2.35, you will work through this generalization (and its consequences) for sheaves of abelian groups. In the next section of the notes, we will discuss a procedure for sheafification which works directly for sheaves with values in an arbitrary algebraic category $\mathcal{D}$.

Exercises

Exercise 1.2.1. Prove Proposition 1.2.1.

Exercise 1.2.2. Show that the functor $\mathcal{F}$ in Example 1.2.2 is a well-defined subfunctor of $\mathrm{Hom}(-, S^1)$ which satisfies the sheaf condition. If needed, also prove the point-set topology lemma used in that proof: if $f \circ g$ is a quotient map, then f is a quotient map.

Exercise 1.2.3. Determine which of the presheaves in Example 1.2.6 are sheaves.

Exercise 1.2.4. Let X and Y_1 be topological spaces with $X \neq \varnothing$. Let Y_2 be a set. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the presheaves on X defined by $\mathcal{F}_1(U) = \mathrm{Hom}_{\mathsf{Top}}(U, Y_1)$ and $\mathcal{F}_2(U) = Y_2$. Show that $\mathcal{F}_1$ and $\mathcal{F}_2$ are not isomorphic unless $|Y_1| = |Y_2| = 1$.

Exercise 1.2.5. Let X be a smooth manifold. Let C^∞_X denote the sheaf of smooth functions on X, i.e. $C^\infty_X(U) = \mathrm{Hom}_{\mathsf{Man}}(U, \mathbb{R})$ for each open set $U \subseteq X$. Show that a continuous function $f : Y \to X$ from a smooth manifold Y is smooth if and only if the composite

$$Y \xrightarrow{f} X \xrightarrow{\varphi} \mathbb{R}$$

is smooth for all global sections $\varphi \in C^\infty_X(X)$.

In this way, C^∞_X determines the smooth structure of X.

Exercise 1.2.6. Show that the functor $\mathcal{S}_\pi$ defined in Example 1.2.9 is indeed a well-defined sheaf.

Exercise 1.2.7. Show that $\mathsf{Sh}(\mathrm{pt}) \simeq \mathsf{Set}$, but $\mathsf{PSh}(\mathrm{pt}) \not\simeq \mathsf{Set}$.

Exercise 1.2.8. Let $\mathcal{C}$ be a category admitting finite products, and fix a terminal object $1 \in \mathcal{C}$. An *(abelian) group object* in $\mathcal{C}$ is an object $X \in \mathcal{C}$ together with morphisms

$$m : X \times X \to X \qquad e : 1 \to X \qquad i : X \to X$$

encoding multiplication, identity, and inverse, satisfying the usual axioms for an (abelian) group. The category of (abelian) group objects in $\mathcal{C}$ will be denoted $\mathsf{Grp}(\mathcal{C})$ (resp. $\mathsf{Ab}(\mathcal{C})$) – its objects are (abelian) group objects in $\mathcal{C}$, and its morphisms are morphisms in $\mathcal{C}$ that respect all of the structure maps.

(a) Make sure this definition makes sense to you, i.e. you can formalize all the missing details. Convince yourself that $\mathsf{Grp}(\mathsf{Set}) \simeq \mathsf{Grp}$ and $\mathsf{Ab}(\mathsf{Set}) \simeq \mathsf{Ab}$.

(b) Let X be a topological space. Show that $\mathsf{Sh}(X, \mathsf{Grp}) \simeq \mathsf{Grp}(\mathsf{Sh}(X))$ and $\mathsf{Sh}(X, \mathsf{Ab}) \simeq \mathsf{Ab}(\mathsf{Sh}(X))$.

(c) Convince yourself that a similar statement holds for Ring and CRing.

(d) For experts: formulate and prove the analogous statement for any locally presentable category.

Exercise 1.2.9. Prove Proposition 1.2.10.

Exercise 1.2.10. For experts: prove Proposition 1.2.11.

Exercise 1.2.11. If you did not do the previous exercise: let $f : X \to Y$ be a continuous function, and let $\mathcal{D}$ be an algebraic category. Show directly that $f^{-1}_{\mathrm{pre}} : \mathsf{PSh}(Y, \mathcal{D}) \to \mathsf{PSh}(X, \mathcal{D})$ is left adjoint to $f_* : \mathsf{PSh}(X, \mathcal{D}) \to \mathsf{PSh}(Y, \mathcal{D})$.

Exercise 1.2.12. Prove Proposition 1.2.16.

Exercise 1.2.13. Let $\pi : E \to X$ be a covering space. Show that, for all $x \in X$, the stalk $(\mathcal{S}_\pi)_x$ is in bijection with the fiber $\pi^{-1}(x)$.

Exercise 1.2.14. Give an example of two sheaves on a space X which are not isomorphic, but which have isomorphic stalks at every point of X.

Exercise 1.2.15. Let $f : X \to Y$ be a continuous map of spaces. Show that $f^{-1}_{\text{pre}} : \mathsf{PSh}(Y, \mathcal{D}) \to \mathsf{PSh}(X, \mathcal{D})$ preserves stalks, i.e. for all $x \in X$ and $\mathcal{F} \in \mathsf{PSh}(Y, \mathcal{D})$, we have a canonical isomorphism

$$(f^{-1}_{\text{pre}}\mathcal{F})_x \cong \mathcal{F}_{f(x)}.$$

Exercise 1.2.16. Prove parts (i) and (ii) of Theorem 1.2.17.

Exercise 1.2.17. Let $f : \{0, 1\} \to \text{pt}$. Show that $f^{-1}_{\text{pre}} : \mathsf{PSh}(\text{pt}) \to \mathsf{PSh}(\{0, 1\})$ does not preserve sheaves.

Exercise 1.2.18. For experts: develop the theory of sheaves, stalks, etc., and prove Theorem 1.2.17 for sheaves with values in an arbitrary locally finitely presentable category. Or, if you prefer, you can do the category of algebras of a finitary monad over Set.

Exercise 1.2.19. In this exercise, you will show that Theorem 1.2.17(iv) can fail badly for sheaves valued in a category which is not algebraic, even one which is complete and cocomplete. We take $\mathcal{D} = \mathsf{Top}$, and read Definition 1.2.8 verbatim, with the equalizer taken in Top. That is: $\mathcal{F} \in \mathsf{PSh}(X, \mathsf{Top})$ is a sheaf if, for every open set $U \subseteq X$ and every open cover $\{U_\alpha\}_\alpha$ of U, the map

$$\mathcal{F}(U) \to \prod_\alpha \mathcal{F}(U_\alpha)$$

is a topological embedding whose image is exactly the set of tuples agreeing on pairwise intersections.

For an open set $U \subseteq \mathbb{R}$, write $C(U)$ for the set of continuous functions $U \to \mathbb{R}$. Define presheaves $\mathcal{F}_k, \mathcal{F}_p \in \mathsf{PSh}(\mathbb{R}, \mathsf{Top})$ by letting $\mathcal{F}_k(U)$ be $C(U)$ with the compact-open topology, letting $\mathcal{F}_p(U)$ be $C(U)$ with the topology of pointwise convergence, and taking the usual restriction maps in both cases.

(a) Show that $\mathcal{F}_p$ is a sheaf.
Recall: The topology of pointwise convergence on $C(U)$ is the initial topology for the collection of evaluation maps $\{\text{ev}_y : C(U) \to \mathbb{R}\}_{y \in U}$.

(b) Show that $\mathcal{F}_k$ is a sheaf.
Hint: If $K \subseteq U$ is compact and $\{U_\alpha\}_\alpha$ covers U, then $K = K_1 \cup \cdots \cup K_n$ for finitely many compact sets K_i, each contained in some U_{α_i}.

(c) Show that the identity maps $C(U) \to C(U)$ assemble into a morphism $\varphi : \mathcal{F}_k \to \mathcal{F}_p$ in $\mathsf{Sh}(\mathbb{R}, \mathsf{Top})$, and that φ is not an isomorphism.

(d) Fix $x \in \mathbb{R}$ and let $\mathcal{G}_x$ denote the set of germs at x of continuous real-valued functions defined near x. Show that $(\mathcal{F}_k)_x$ and $(\mathcal{F}_p)_x$ both have underlying set $\mathcal{G}_x$, and that in both cases the topology on the stalk is the initial topology for the map $\mathcal{G}_x \to \mathbb{R}$ sending a germ to its value at x. Conclude that φ_x is a homeomorphism for every $x \in \mathbb{R}$.

(e) Conclude that φ is a morphism of sheaves which is an isomorphism stalkwise, but is not an isomorphism.

Exercise 1.2.20. Prove Proposition 1.2.26.

Exercise 1.2.21. Give an example of a space X and a basis $\mathcal{B}$ for its topology such that the restriction functor $\mathsf{PSh}(X, \mathcal{D}) \to \mathsf{PSh}(\mathcal{B}, \mathcal{D})$ is not fully faithful.

Exercise 1.2.22. Prove the claim in Remark 1.2.31.

Exercise 1.2.23. Show that $\widetilde{\mathcal{F}}$ in Theorem 1.2.29 is indeed a sheaf on X.

Exercise 1.2.24. A *ringed space* is a pair $(X, \mathcal{O}_X)$ of a topological space X and a sheaf $\mathcal{O}_X \in \mathsf{Sh}(X, \mathsf{CRing})$, called the *structure sheaf*. A *morphism of ringed spaces* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^\sharp)$ consisting of a continuous map $f : X \to Y$ together with a morphism $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y.

For M a smooth manifold, write C^∞_M for the sheaf $\mathrm{Hom}_{\mathsf{Man}}(-, \mathbb{R})$ of smooth functions on M as in Exercise 1.2.5; note that (M, C^∞_M) is a ringed space.

Let us call a ringed space $(X, \mathcal{O}_X)$ an *n-dimensional smooth space* if X is Hausdorff and second countable[24], and every point of X admits an open neighborhood U for which there is an isomorphism of ringed spaces

$$(U, \mathcal{O}_X|_U) \cong (\mathbb{R}^n, C^\infty_{\mathbb{R}^n}).$$

A *smooth space* is an n-dimensional smooth space for some $n \geq 0$. A morphism of smooth spaces is simply a morphism of ringed spaces.

In this exercise, you will show that the category of smooth spaces is equivalent to the category of smooth manifolds. The upshot is that this gives a definition of smooth manifolds (not necessarily embedded in $\mathbb{R}^n$!) which does *not* require explicitly recording/choosing charts and atlases.

(a) Let X be a smooth n-manifold.

  (i) Show that (X, C^∞_X) is an n-dimensional smooth space.

  (ii) Show that C^∞_X is a sheaf of $\mathbb{R}$-algebras[25] in a unique way.
  Hint: Use the fact that the only ring endomorphism of $\mathbb{R}$ is the identity.

(b) Let $(f, f^\sharp) : (\mathbb{R}^n, C^\infty_{\mathbb{R}^n}) \to (\mathbb{R}^m, C^\infty_{\mathbb{R}^m})$ be a morphism of ringed spaces.

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[24] We also make these point-set assumptions about smooth manifolds.

[25] You may add the category of $\mathbb{R}$-algebras to the list of "algebraic categories". Alternatively, a sheaf of $\mathbb{R}$-algebras can be defined as a sheaf $\mathcal{F}$ of commutative rings together with a morphism $\underline{\mathbb{R}} \to \mathcal{F}$, where $\underline{\mathbb{R}}$ denotes the constant presheaf of commutative rings with value $\mathbb{R}$.

(i) Show that $f^\sharp$ is a morphism of sheaves of $\mathbb{R}$-algebras.

(ii) For all $V \in \mathsf{Open}(\mathbb{R}^m)$ and all $\varphi \in C^\infty_{\mathbb{R}^m}(V)$, show that

$$f^\sharp_V(\varphi) = \varphi \circ f|_{f^{-1}(V)}.$$

Hint: Let $x \in f^{-1}(V)$ and $s \in \mathbb{R}$ with $s \neq \varphi(f(x))$. It suffices to show that $(f^\sharp_V \varphi)(x) \neq s$. Use the fact that $\varphi - s$ is nonzero on some open neighborhood of $f(x)$.

(c) Let $f : X \to Y$ be a smooth map between smooth manifolds. Show that there is a unique morphism $f^\sharp : C^\infty_Y \to f_* C^\infty_X$ of sheaves of rings on Y, and that $f^\sharp$ happens to be a morphism of sheaves of $\mathbb{R}$-algebras.

(d) Let $(X, \mathcal{O}_X)$ be an n-dimensional smooth space. Show that X has a unique smooth n-manifold structure such that:

($\star$) For every open set $U \subseteq X$ and every isomorphism of ringed spaces $(\mathbb{R}^n, C^\infty_{\mathbb{R}^n}) \xrightarrow{\cong} (U, \mathcal{O}_X|_U)$, the underlying map $\mathbb{R}^n \to U$ is a diffeomorphism.

Then, show that $\mathcal{O}_X \cong C^\infty_X$ as sheaves of rings.

(e) Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of smooth spaces. Show that $f : X \to Y$ is smooth, where we view X and Y as smooth manifolds by (d). Conclude that the category of smooth manifolds is equivalent to the category of smooth spaces.

Exercise 1.2.25. Let $\mathcal{F}$ be a presheaf of sets on a space X. Show that the canonical projection $\pi : E_{\mathcal{F}} \to X$ is continuous.

Exercise 1.2.26. Let $p : E \to X$ be a covering space, i.e. there exists a discrete space F such that for all $x \in X$, there is an open neighborhood $U \subseteq X$ of x and a homeomorphism $p^{-1}(U) \cong U \times F$ such that

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "p|_{p^{-1}(U)}"'] & & U \times F \arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$

commutes, where π is the canonical projection. Show that p is a local homeomorphism.

Exercise 1.2.27. Let $f : X \to Y$ be a local homeomorphism, and let $g : Z \to Y$ be a continuous map. Show that the pullback $f' : X \times_Y Z \to Z$ is a local homeomorphism.

Exercise 1.2.28. Prove Proposition 1.2.39. That is: let $f : E \to X$ and $g : E' \to X$ be local homeomorphisms, and let $\varphi : E \to E'$ be a continuous map with $g \circ \varphi = f$. Show that φ is itself a local homeomorphism.

Exercise 1.2.29. Complete the proof of Proposition 1.2.40.

Exercise 1.2.30. Complete the proof of Proposition 1.2.41.

Exercise 1.2.31. Show that the unit map $\eta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^{++}$ is natural in $\mathcal{F} \in \mathsf{PSh}(X)$.

Exercise 1.2.32. Show that the map β constructed in the proof of Theorem 1.2.45 is natural in $p \in \mathsf{Top}^{\mathrm{lh}}/X$.

Exercise 1.2.33. A *constant sheaf* on a space X is a sheaf $\mathcal{F}$ such that there exists a set S with $\mathcal{F} \cong \underline{S}^{++}$, where $\underline{S}$ is the constant presheaf with value S. Show that $\underline{S}$ is a sheaf on X if and only if $|S| = 1$.

Exercise 1.2.34. Show that the sheafification functor $\mathsf{PSh}(X) \to \mathsf{Sh}(X)$ preserves finite limits. Deduce that $f^{-1} : \mathsf{Sh}(Y) \to \mathsf{Sh}(X)$ preserves finite limits for any continuous map $f : X \to Y$.

Exercise 1.2.35. From Exercise 1.2.8, we know that $\mathsf{Sh}(X, \mathsf{Ab}) \simeq \mathsf{Ab}(\mathsf{Sh}(X))$. By Theorem 1.2.45, we have $\mathsf{Sh}(X, \mathsf{Ab}) \simeq \mathsf{Ab}(\mathsf{Top}^{\mathrm{lh}}/X)$.

(a) Describe concretely what an abelian group object in $\mathsf{Top}^{\mathrm{lh}}/X$ is. Show that each fiber of an abelian group object in $\mathsf{Top}^{\mathrm{lh}}/X$ is an abelian group in a natural way, and this computes the stalks of the corresponding sheaf of abelian groups.

(b) Show that the sheafification functor $\mathsf{PSh}(X) \to \mathsf{Sh}(X)$ preserves abelian group objects (make precise what this means), and deduce that there is an induced functor $\mathsf{PSh}(X, \mathsf{Ab}) \to \mathsf{Sh}(X, \mathsf{Ab})$. Show that this functor is left adjoint to the inclusion $\mathsf{Sh}(X, \mathsf{Ab}) \hookrightarrow \mathsf{PSh}(X, \mathsf{Ab})$.

(c) Show that the sheafification functor $\mathsf{PSh}(X, \mathsf{Ab}) \to \mathsf{Sh}(X, \mathsf{Ab})$ preserves finite limits.

(d) Show that $\mathsf{PSh}(X, \mathsf{Ab})$ and $\mathsf{Sh}(X, \mathsf{Ab})$ are both abelian categories. Then show that:

  (i) Exactness in $\mathsf{PSh}(X, \mathsf{Ab})$ is checked on local sections, i.e. a sequence $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ in $\mathsf{PSh}(X, \mathsf{Ab})$ is exact if and only if for every open set $U \subseteq X$, the sequence $\mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$ is exact in Ab.

  (ii) Exactness in $\mathsf{Sh}(X, \mathsf{Ab})$ is checked on stalks, i.e. a sequence $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ in $\mathsf{Sh}(X, \mathsf{Ab})$ is exact if and only if for every point $x \in X$, the sequence $\mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x$ is exact in Ab.

  (iii) The inclusion $\mathsf{Sh}(X, \mathsf{Ab}) \hookrightarrow \mathsf{PSh}(X, \mathsf{Ab})$ is left exact, but not right exact in general.

  (iv) The sheafification functor $\mathsf{PSh}(X, \mathsf{Ab}) \to \mathsf{Sh}(X, \mathsf{Ab})$ is exact.

  (v) Let $f : X \to Y$ be a continuous map. Show that $f_* : \mathsf{Sh}(X) \to \mathsf{Sh}(Y)$ preserves abelian group objects, and thus induces a map $f_* : \mathsf{Sh}(X, \mathsf{Ab}) \to \mathsf{Sh}(Y, \mathsf{Ab})$. Show that $f_* : \mathsf{Sh}(X, \mathsf{Ab}) \to \mathsf{Sh}(Y, \mathsf{Ab})$ is left exact, but not right exact in general.

(vi) Let $f : X \to Y$ be a continuous map. Show that $f^{-1} : \mathsf{Sh}(Y) \to \mathsf{Sh}(X)$ preserves abelian group objects, and thus induces a map $f^{-1} : \mathsf{Sh}(Y, \mathsf{Ab}) \to \mathsf{Sh}(X, \mathsf{Ab})$ which is left adjoints to f_*. Show that $f^{-1} : \mathsf{Sh}(Y, \mathsf{Ab}) \to \mathsf{Sh}(X, \mathsf{Ab})$ is exact.

(vii) Let $f : X \to Y$ be a local homeomorphism. Show that $f_! : \mathsf{Sh}(X) \to \mathsf{Sh}(Y)$ does *not* lift to a functor on abelian group objects. Nevertheless, show that $f^{-1} : \mathsf{Sh}(Y, \mathsf{Ab}) \to \mathsf{Sh}(X, \mathsf{Ab})$ admits a left adjoint $f_!^{\mathsf{Ab}} : \mathsf{Sh}(X, \mathsf{Ab}) \to \mathsf{Sh}(Y, \mathsf{Ab})$, which is exact. Conclude that f^{-1} preserves injective objects.

1.3 Sites

We began the course by discussing "sheaves on the category of sets", which is certainly not a special case of $\mathsf{Sh}(X)$ for X a topological space, since Set is not equivalent to $\mathsf{Open}(X)$ for any topological space X. In this section, we will discuss a generalization of the notion of sheaf which encompasses both sheaves on topological spaces and sheaves on the category of sets. For this, we must replace $\mathsf{Open}(X)$ and Set with a more general datum called a *site*. A site is a category together with a collection of "covering families" – these covering families specify exactly which equalizers should appear in the sheaf condition. It will turn out that every category can be equipped with a canonical choice of covering families, and that sheaves on Set or $\mathsf{Open}(X)$ with respect to these canonical covering families are exactly the sheaves we have already discussed.

1.3.1 Grothendieck Topologies and the Sheaf Condition

Definition 1.3.1. Let $\mathcal{C}$ be a category. A *sieve* on an object $x \in \mathcal{C}$ is a subfunctor $S \subseteq y(x)$ of the representable presheaf $y(x) = \mathrm{Hom}_{\mathcal{C}}(-, x)$. That is, for all objects $c \in \mathcal{C}$, we have a subset $S(c) \subseteq \mathrm{Hom}_{\mathcal{C}}(c, x)$, and for all morphisms $f : c' \to c$ in $\mathcal{C}$, we have $g \circ f \in S(c')$ for all $g \in S(c)$.

⚠ **Warning.** If $\mathcal{C}$ is a large category, then $\mathsf{PSh}(\mathcal{C})$ is not necessarily a category in the usual sense (its hom-sets may in fact be proper classes). However, the notions of presheaf and subfunctor still make sense, so in particular it is fine to talk about sieves. I will try to be careful about size issues in what follows, but we'll be forced to return to this technical point again soon, and it is always worth thinking about which things are "small" and "large"! This is a spot where the theory of Grothendieck universes would be very useful to make things precise, but I don't want to assume knowledge of Grothendieck universes in this course. I've included an appendix Axiom A.2.3 which introduces Grothendieck universes, but you can safely ignore it.

A sieve is, in other words, a collection of morphisms into x which is closed under precomposition. It's clear that the collection of sieves on x is closed under arbitrary intersection (the empty intersection is the maximum sieve $y(x)$). Thus, we may talk about the sieve on x generated by a collection of morphisms $\{f_\alpha\}_\alpha$

with codomain x, which is the smallest sieve containing all of the f_α. Explicitly, the sieve generated by $\{f_\alpha\}_\alpha$ consists of all morphisms which factor through some f_α, i.e.

$$S(c) = \{g \in \mathcal{C}(c, x) : g = f_\alpha \circ h \text{ for some } \alpha \text{ and some } h : c \to \mathrm{dom}(f_\alpha)\}.$$

Example 1.3.2. Let X be a topological space, and let $U \subseteq X$ be an open set. Let $\{U_\alpha\}_\alpha$ be an open cover of U. The sieve on $U \in \mathsf{Open}(X)$ generated by the inclusions $U_\alpha \hookrightarrow U$ is the subfunctor $S \subseteq y(U)$ defined by

$$S(V) = \{V \hookrightarrow U : V \subseteq U_\alpha \text{ for some } \alpha\}.$$

If S is a sieve on x and $f : x' \to x$ is a morphism in $\mathcal{C}$, we may define the *pullback sieve* f^*S on x' by

$$(f^*S)(z) = \{g \in \mathcal{C}(z, x') : f \circ g \in S(z)\}.$$

Example 1.3.3. Let X be a topological space, let $U \subseteq X$ be an open set, and let $\{U_\alpha\}_\alpha$ be an open cover of U. Let $S \subseteq y(U)$ be the sieve generated by the inclusions $U_\alpha \hookrightarrow U$. Let $V \subseteq U$ be an open subset. Then the pullback sieve V^*S on V is the sieve generated by the inclusions $V \cap U_\alpha \hookrightarrow V$, i.e.

$$(V^*S)(W) = \{W \hookrightarrow V : W \subseteq V \cap U_\alpha \text{ for some } \alpha\}.$$

Definition 1.3.4. Let $\mathcal{C}$ be a category. A *Grothendieck topology* $\mathcal{J}$ on $\mathcal{C}$ is a choice of a collection[26] of sieves $\mathcal{J}(x)$ on each object $x \in \mathcal{C}$, called the *covering sieves*, such that:

(T1) For all $S \in \mathcal{J}(x)$ and all morphisms $f : y \to x$, $f^*S \in \mathcal{J}(y)$.

(T2) For all $S \in \mathcal{J}(x)$ and all sieves T on x, if $f^*T \in \mathcal{J}(y)$ for all $f : y \to x$ in S, then $T \in \mathcal{J}(x)$.

(T3) $y(x) \in \mathcal{J}(x)$ for all $x \in \mathcal{C}$.

A *site* is a pair $(\mathcal{C}, \mathcal{J})$ of a category $\mathcal{C}$ and a Grothendieck topology $\mathcal{J}$ on $\mathcal{C}$.

Example 1.3.5. Let X be a topological space. There is a Grothendieck topology $\mathcal{J}$ on $\mathsf{Open}(X)$ defined by declaring the covering sieves on U to be precisely the sieves generated by open covers of U, as in Example 1.3.2.

Example 1.3.6. There is a Grothendieck topology on Set defined by declaring the covering sieves $S \in \mathcal{J}(X)$ to be precisely those for which the morphisms in S are jointly surjective, i.e. for all $x \in X$ there exists some $f : Y \to X$ in $S(Y)$ and some $y \in Y$ such that $f(y) = x$.

Remark 1.3.7. One sometimes sees sites defined in a slightly different way, in terms of *Grothendieck pre-topologies*. Here, instead of specifying covering sieves, one specifies covering families $\{f_\alpha : x_\alpha \to x\}_\alpha$ of morphisms with codomain x. These covering families are required to satisfy the following axioms:

[26] Potentially a proper class!

(PT1) For all covering families $\{f_\alpha : x_\alpha \to x\}_\alpha$ and all morphisms $g : y \to x$, $\{x_\alpha \times_x y \to y\}_\alpha$ is a covering family.

(PT2) For all covering families $\{f_\alpha : x_\alpha \to x\}_\alpha$ and all collections of covering families $\{g_{\alpha,\beta} : x_{\alpha,\beta} \to x_\alpha\}_\beta$, the family $\{f_\alpha \circ g_{\alpha,\beta} : x_{\alpha,\beta} \to x\}_{\alpha,\beta}$ is a covering family.

(PT3) The singleton family $\{\mathrm{id}_x : x \to x\}$ is a covering family for all x.

Any Grothendieck pre-topology generates a Grothendieck topology by declaring a sieve S on x to be covering if and only if it contains a covering family[27]. There are two downsides to this approach:

- Many distinct pre-topologies may generate the same topology, and the eventual definition of "sheaves on a site" that we will use only depends on the Grothendieck topology, not the pre-topology.
- The axioms of a Grothendieck pre-topology assume the existence of (at least some) pullbacks in $\mathcal{C}$, while the axioms of a Grothendieck topology do not.

Suppose for a moment that $\mathcal{C}$ is small. The Yoneda lemma says that

$$\mathcal{F}(x) \cong \mathrm{Hom}_{\mathsf{PSh}(\mathcal{C})}(y(x), \mathcal{F})$$

for all presheaves $\mathcal{F} \in \mathsf{PSh}(\mathcal{C})$. Consequently, for any sieve $S \subseteq y(x)$, we have a canonical map

$$\mathcal{F}(x) \cong \mathrm{Hom}_{\mathsf{PSh}(\mathcal{C})}(y(x), \mathcal{F}) \to \mathrm{Hom}_{\mathsf{PSh}(\mathcal{C})}(S, \mathcal{F}).$$

We say that $\mathcal{F}$ *satisfies the sheaf condition with respect to* S or *is local with respect to* S if this map is a bijection. We will write $\mathcal{F}(S)$ as shorthand for $\mathrm{Hom}_{\mathsf{PSh}(\mathcal{C})}(S, \mathcal{F})$.

If $\mathcal{C}$ is not a small category, the version of the Yoneda lemma we have stated (Proposition 1.2.5) does not apply. However, much of it still holds:

- It is still true that $\mathrm{Hom}(y(x), \mathcal{F}) \cong \mathcal{F}(x)$ naturally in presheaves $\mathcal{F}$ and objects x; in particular, $\mathrm{Hom}(y(x), \mathcal{F})$ is a set.
- It is still true that every presheaf $\mathcal{F}$ is a colimit of representable presheaves in a canonical way (this colimit is indexed by a potentially large category, but it does exist).

In this case, we can still obtain a canonical map $\mathcal{F}(x) \to \mathcal{F}(S) := \mathrm{Hom}(S, \mathcal{F})$ for any sieve S on x, but $\mathcal{F}(S)$ may be a proper class. Still, it makes sense to ask whether or not $\mathcal{F}$ satisfies the sheaf condition with respect to S!

If S is generated by a set of morphisms, then we can guarantee that $\mathcal{F}(S)$ is a set. Many sites encountered in practice have the property that all covering sieves are generated by sets.

[27] It is worth taking the time to understand how the axioms PT1-PT3 translate to the axioms T1-T3 via this construction!

Proposition 1.3.8. *Let S be a sieve on $x \in \mathcal{C}$. If S is generated by a set of morphisms $\{f_i : c_i \to x\}_i$, then*

$$\mathcal{F}(S) := \mathrm{Hom}(S, \mathcal{F})$$

is a set for any presheaf $\mathcal{F} \in \mathsf{PSh}(\mathcal{C})$.

Proof. The key is that, in this case, the morphism

$$\coprod_i y(c_i) \to S$$

determined by the elements $f_i \in S(c_i)$ is an epimorphism in $\mathsf{PSh}(\mathcal{C})$; i.e. it is a natural transformation whose components are all surjections. To see this, note that the image of this map $\coprod_i y(c_i) \to S$ is a well-defined subfunctor of S (thus in particular a sieve on x) which contains all of the f_i; since S is the smallest sieve containing all of the f_i, it follows that this image must be all of S. It follows that the induced map

$$\mathcal{F}(S) = \mathrm{Hom}(S, \mathcal{F}) \to \mathrm{Hom}(\coprod_i y(c_i), \mathcal{F}) \cong \prod_i \mathrm{Hom}(y(c_i), \mathcal{F}) \cong \prod_i \mathcal{F}(c_i)$$

is injective, and thus $\mathrm{Hom}(S, \mathcal{F})$ is a set. $\square$

Proposition 1.3.9. *Let $\mathcal{C}$ be a category, and let S be a sieve on $x \in \mathcal{C}$. Let S/x denote the full subcategory of $\mathcal{C}/x$ spanned by the morphisms in S.*

Then, for any presheaf $\mathcal{F}$, elements of $\mathcal{F}(S)$ are in bijection with compatible families $(s_f)_{f \in S}$, where $s_f \in \mathcal{F}(c)$ for each $f : c \to x$ in S. Here compatibility means that for all morphisms $g : f' \to f$ in S/x, we have $\mathcal{F}(g)(s_f) = s_{f'}$.

Proof. Given $\alpha : S \to \mathcal{F}$, set $s_f := \alpha_c(f) \in \mathcal{F}(c)$ for each $f : c \to x$ in S. This is compatible by naturality of α. In the other direction, given a compatible family $(s_f)_{f \in S}$, define $\alpha_c : S(c) \to \mathcal{F}(c)$ by $\alpha_c(f) := s_f$. These are the components of a natural transformation $\alpha : S \to \mathcal{F}$ by the compatibility condition. $\square$

This is almost just saying that $\mathcal{F}(S)$ is the limit of the diagram

$$(\textstyle\int S)^{\mathrm{op}} \cong (S/x)^{\mathrm{op}} \xrightarrow{\mathrm{dom}} \mathcal{C}^{\mathrm{op}} \xrightarrow{\mathcal{F}} \mathsf{Set},$$

coming from the fact that S is the colimit of the diagram

$$\textstyle\int S \cong S/x \xrightarrow{\mathrm{dom}} \mathcal{C} \xrightarrow{y} \mathsf{PSh}(\mathcal{C}).$$

The only trouble is that the indexing category $\int S$ is potentially large, and so the desired limit does not necessarily exist in Set, as we have discussed. However, the colimit defining S does exist in PSh, and we can describe directly what elements of $\mathcal{F}(S) = \mathrm{Hom}(S, \mathcal{F})$ are by the universal property of colimits. This gives the statement in the proposition.

Proposition 1.3.10. *Let $\mathcal{C}$ be a category, and let S be a sieve on $x \in \mathcal{C}$. Assume that the slice category $\mathcal{C}/x$ admits binary products*[28] *and that S is generated by a set of morphisms $\{f_i : c_i \to x\}_i$. Then $\mathcal{F}(S)$ is naturally isomorphic to the equalizer of the diagram*

$$\prod_i \mathcal{F}(c_i) \rightrightarrows \prod_{i,j} \mathcal{F}(c_i \times_x c_j).$$

Proof. Recall from the proof of Proposition 1.3.8 that restriction along the f_i gives an injection $\mathcal{F}(S) \hookrightarrow \prod_i \mathcal{F}(c_i)$; we must identify its image with the equalizer, and for this we use the description of Proposition 1.3.9.

If $(s_f)_{f\in S}$ is a compatible family, then for all i, j we have $f_i \circ p_1 = f_j \circ p_2$, where $p_1 : c_i \times_x c_j \to c_i$ and $p_2 : c_i \times_x c_j \to c_j$ are the canonical projections, so

$$\mathcal{F}(p_1)(s_{f_i}) = s_{f_i \circ p_1} = s_{f_j \circ p_2} = \mathcal{F}(p_2)(s_{f_j}).$$

Conversely, suppose $(s_i)_i$ satisfies $\mathcal{F}(p_1)(s_i) = \mathcal{F}(p_2)(s_j)$ for all i, j. Every $f : d \to x$ in S factors as $f = f_i \circ h$ for some i and some $h : d \to c_i$; set $s_f := \mathcal{F}(h)(s_i)$. This is independent of the factorization: if also $f = f_j \circ k$, then (h, k) induces $r : d \to c_i \times_x c_j$ with $h = p_1 \circ r$ and $k = p_2 \circ r$, whence

$$\mathcal{F}(h)(s_i) = \mathcal{F}(r)(\mathcal{F}(p_1)(s_i)) = \mathcal{F}(r)(\mathcal{F}(p_2)(s_j)) = \mathcal{F}(k)(s_j).$$

The family $(s_f)_{f\in S}$ is compatible: if $g : f' \to f$ is a morphism in S/x and $f = f_i \circ h$, then $f' = f_i \circ (h \circ g)$, so $s_{f'} = \mathcal{F}(h \circ g)(s_i) = \mathcal{F}(g)(s_f)$. Finally, taking $h = \mathrm{id}_{c_i}$ shows $s_{f_i} = s_i$. $\square$

This understanding allows us to extend our sheaf condition so it applies to presheaves with values in any algebraic category.

Definition 1.3.11. Let $\mathcal{C}$ be a category, let S be a sieve on $x \in \mathcal{C}$, and let $\mathcal{D}$ be an algebraic category. A presheaf $\mathcal{F} \in \mathsf{PSh}(\mathcal{C}, \mathcal{D})$ is said to *satisfy the sheaf condition with respect to S* (or be *local with respect to S*) if its underlying Set-valued presheaf (coming from the forgetful functor $\mathcal{D} \to \mathsf{Set}$) is local with respect to S.

We have same equalizer description of the sheaf condition in this more general setting, coming from the fact that limits in $\mathcal{D}$ are computed in Set.

Proposition 1.3.12. *Let $\mathcal{D}$ be an algebraic category. If S is generated by a set of morphisms $\{f_i : c_i \to x\}_i$ and $\mathcal{C}/x$ admits binary products, then $\mathcal{F} \in \mathsf{PSh}(\mathcal{C}, \mathcal{D})$ is local with respect to S if and only if*

$$\mathcal{F}(x) \to \prod_i \mathcal{F}(c_i) \rightrightarrows \prod_{i,j} \mathcal{F}(c_i \times_x c_j)$$

is an equalizer diagram in $\mathcal{D}$.

[28] These are just pullback squares with x in the corner.

This allows us to define sheaves on a site with values in an algebraic category[29].

Definition 1.3.13. Let $(\mathcal{C}, \mathcal{J})$ be a site, and let $\mathcal{D}$ be an algebraic category. A presheaf $\mathcal{F} \in \mathsf{PSh}(\mathcal{C}, \mathcal{D})$ is a *sheaf* if it is local with respect to all covering sieves. We write $\mathsf{Sh}(\mathcal{C}, \mathcal{D})$ for the full subcategory of $\mathsf{PSh}(\mathcal{C}, \mathcal{D})$ spanned by the sheaves.

Again, we warn that $\mathsf{Sh}(\mathcal{C}, \mathcal{D})$ may fail to be a category in the traditional sense if $\mathcal{C}$ is not small (because its hom-sets may be too large).

Exercises

Exercise 1.3.1. Show that the collection of sieves in Example 1.3.5 satisfies the axioms of a Grothendieck topology.

Exercise 1.3.2. Check that Definition 1.3.13 recovers Definition 1.2.8 when $(\mathcal{C}, \mathcal{J}) = (\mathsf{Open}(X), \mathcal{J})$ for a space X, where $\mathcal{J}$ is the topology defined in Example 1.3.5.

Exercise 1.3.3. For experts: if $\mathcal{D}$ is an arbitrary category, we say that a presheaf $\mathcal{F} \in \mathsf{PSh}(\mathcal{C}, \mathcal{D})$ is a sheaf if the Set-valued presheaf

$$\mathrm{Hom}_{\mathcal{D}}(d, \mathcal{F}(-)) : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$$

is a sheaf for all objects $d \in \mathcal{D}$. Show that this recovers Definition 1.3.13 when $\mathcal{D}$ is an algebraic category.

[29] Or, as before, any locally finitely presentable category, if you're willing to be more abstract.

Appendix A

Size Issues in Category Theory

In an introduction to category theory, one often learns that a category $\mathcal{C}$ consists of a *collection* of objects (together with morphisms, composition operations, etc. satisfying some axioms). Often in mathematics, the word "collection" is synonymous with the word "set", but not here! What is meant precisely by the word "collection" in the definition of a category depends slightly on how one wishes to formalize their set-theoretic foundations for category theory, and there are generally two popular approaches here. We will work in ZFC set theory.

A.1 Classes

A famous argument of Russell shows that there is no set of all sets, i.e.

$$\nexists A \forall x (x \in A). \tag{A.1}$$

We quickly recount the proof.

Proof. Suppose (towards a contradiction) that the set A of all sets did exist, i.e. $\exists A \forall x (x \in A)$. Then one could form the set

$$B := \{x \in A : x \notin x\}.$$

Now by construction (and since $B \in A$), we have $B \in B \iff B \notin B$, which is a contradiction. $\square$

We emphasize that (A.1) is a statement in the language of set theory, and so its proof relies on some axioms of set theory. In this case, the proof of (A.1) relies on only one axiom (schema), the *axiom schema of restricted comprehension*.

Axiom A.1.1 (Axiom Schema of Restricted Comprehension[1]). Let $\varphi(x)$ be a formula in the language of set theory (i.e. φ may have x as a free variable and no other free variables).

For all A, there exists B such that for all x, $x \in B$ if and only if $x \in A$ and $\varphi(x)$ holds.

This is called an axiom *schema* because it is actually an infinite collection of axioms (a different axiom for each formula φ). It may be tempting to try to rewrite this axiom schema as a single axiom of the form $\forall\varphi(\cdots)$, but this does not make sense – axioms of set theory must be statements in the language of set theory. Formulas in the language of set theory are not themselves sets, so we may not quantify over formulas!

Despite its name, the axiom schema of restricted comprehension is easy to understand. In plain English: given any set A and any proposition $\varphi(x)$, one may construct a set consisting precisely of those elements of A which satisfy the proposition φ. If one also has the axiom of extensionality[2], which one should always include in their axioms of set theory, then the set B described above is necessarily unique, and we write

$$\{x \in A : \varphi(x)\}$$

to denote this set. Indeed, the axiom schema of restricted comprehension is what gives meaning to this set-builder notation! If you have ever defined a set like

$$\{x \in \mathbb{R} : |x| < 1\},$$

you were using the axiom schema of restricted comprehension. The tool of set-builder notation is crucial for developing almost any piece of mathematics, and so the axiom schema of restricted comprehension is essential in any useful theory of sets.

Now the key point is that the axiom schema of restricted comprehension allows us to build *subsets* $\{x \in A : \varphi(x)\}$ of any *pre-existing set* A, but it does *not* allow us to construct new sets which are not subsets of a pre-existing set! In particular, the axiom schema does **not** allow us to interpret

$$\{x : x \notin x\}$$

(for example) as a well-defined set[3]. Indeed, as we have seen, the axiom schema of restricted comprehension allows us to prove that no such set exists! On the other hand, for any *given* set A, one *may* form

$$A_\top \coloneqq \{x \in A : x \notin x\},$$

[1] This is actually weaker than the standard axiom schema of restricted comprehension used in (e.g.) ZFC set theory, but the details are unimportant here.

[2] The axiom of extensionality states that sets are determined by their elements, i.e. $\forall A \forall B (A = B \iff \forall x (x \in A \iff x \in B))$. We will assume this axiom always, though the proof of (A.1) given above does not actually require the axiom of extensionality.

[3] This expression is an example of an *unrestricted* comprehension, which our axiom says nothing about.

i.e. the set of all elements of A which do not contain themselves[4]. This is simply the set whose existence is guaranteed by the axiom schema of restricted comprehension applied to the formula $\varphi(x)$ given by "$x \notin x$". For a more extreme example, one may not interpret

$$\{x : x = x\}$$

as a set (it would be the "set of all sets", which we have shown does not exist), but one may form the set $\{x \in A : x = x\}$ given any set A (which will always be equal to A).

On the other hand, we would like to have access to the category of sets, which we will denote Set. Somehow, every set should be an object of Set, i.e. the collection of objects of Set should be the collection of all sets (whatever that means). We know that there is no set of all sets, so this "collection of objects" must not be a set, but instead something else.

Definition A.1.2. A *class* is an equivalence class of formulas (in the language of set theory) with one free variable, with respect to logical equivalence. We say that a formula $\varphi(x)$ is logically equivalent to another formula $\psi(x)$ if and only if

$$\forall x(\varphi(x) \iff \psi(x))$$

holds. We may write $\{x : \varphi(x)\}$ to denote the equivalence class of a formula $\varphi(x)$.

For any formula $\varphi(x)$ and any set y, we may write $y \in \{x : \varphi(x)\}$ to mean $\varphi(y)$, which we note is well-defined by the definition of logical equivalence of formulas.

Now we have a way (tautologically) to interpret

$$\{x : x = x\}$$

as a *class*, and we call this class the *class of all sets*. Now it is easy to prove the statement

$$\forall y(y \in \{x : x = x\})$$

since this is just shorthand for $\forall y(y = y)$, which is true! Simply put, classes are *unrestricted* comprehensions, and they behave just like sets (for any given set S and any given class C, the statement $S \in C$ is either true or false), except that they are not sets (e.g. one may not quantify over classes – the variables appearing in a formula in the language of set theory are sets!). For example, we have a class $C = \{x : x \notin x\}$, but we cannot consider the statement "$C \in C$" to derive a contradiction: it does not make sense to ask if one class is an element of another! The elements of a class are sets.

Now consider the class $\{x : x \neq x\}$. One can just as easily prove that $\forall y(y \notin \{x : x \neq x\})$, since this is shorthand for $\forall y(\neg(y \neq y))$, which is true. In

[4] Using the *foundation* and *pairing* axioms of ZFC, one may prove $\forall x(x \notin x)$, and conclude that $A_\top = A$ for any set A.

other words, we should think of $\{x : x \neq x\}$ as the "empty class". There is also the empty set $\varnothing$[5], and we note that

$$\forall y(y \in \varnothing \iff y \in \{x : x \neq x\})$$

holds. In other words, the class $\{x : x \neq x\}$ has the same elements as the set $\varnothing$! We should therefore formally identify the class $\{x : x \neq x\}$ with the set $\varnothing$, and in general identify any class C with any set S whenever C and S have the same elements.

Convention A.1.3. Let C be a class and let S be a set. If

$$\forall x(x \in C \iff x \in S)$$

holds, we identify C with S and (in particular) say that C *is a set*. If C is not identified with any set S, i.e. if

$$\forall S \exists x(\neg(x \in C \iff x \in S))$$

holds, then we say that C is a *proper class*.

We note that this does not cause any additional identifications: if classes C_1 and C_2 are both identified with a set S, then $\forall x(x \in C_1 \iff x \in C_2)$ holds, so already $C_1 = C_2$. Likewise, if a class C is identified both with a set S_1 and with a set S_2, then $\forall x(x \in S_1 \iff x \in S_2)$ holds, so already $S_1 = S_2$.

We have seen one example of a proper class already, namely the class of all sets $\{x : x = x\}$. There are many others, for example the class of all groups

$$\{x : \exists G \exists \star (x = (G, \star) \text{ and } (G, \star) \text{ is a group})\}. \tag{A.2}$$

Of course, here the text "$(G, \star)$ is a group" stands in for a very long statement in the language of set theory which encodes the group axioms for the pair $(G, \star)$. We might also write (as shorthand)

$$\{(G, \star) : (G, \star) \text{ is a group}\} \tag{A.3}$$

for this class, abusing notation since this is not precisely in the form $\{x : \varphi(x)\}$. It's not a big deal, because there is a canonical procedure to rewrite (A.3) in the standard form (A.2).

In the other direction, we have seen an example of a non-proper class, i.e. a class which is also a set, namely $\{x : x \neq x\}$ (which is equal to $\varnothing$).

We may now make precise the definition of a category. Important for the definition will be the notion of a function between classes.

Definition A.1.4. Let C and C' be classes. We say $C \subseteq C'$ if and only if

$$\forall x(x \in C \implies x \in C').$$

[5] The existence of $\varnothing$ follows (in ZFC) from the axiom of infinity together with the axiom schema of restricted comprehension.

Note that this extends the subset relation between sets, since every set is also (identified with) a class!

Definition A.1.5. Let C and C' be classes. We denote by $C \times C'$ the class

$$\{(x, y) : x \in C \wedge y \in C'\},$$

which (as a reminder) is shorthand for

$$\{p : \exists x \exists y (x \in C \wedge y \in C' \wedge p = (x, y))\}.$$

Definition A.1.6. Let C and C' be classes. By a *function* $C \to C'$ we will mean a class $f \subseteq C \times C'$ such that

$$\forall x (x \in C \implies \exists! y (y \in C' \wedge (x, y) \in f))$$

holds.

Note that when C and C' are sets, this recovers exactly the notion of a function between sets. In any case, when $f : C \to C'$ is a function between classes and $x \in C$ is an element, we denote by $f(x)$ the unique element of C' such that $(x, f(x)) \in f$.

Definition A.1.7 (Categories, Version 1)**.** A category $\mathcal{C}$ consists of the following data:

- A class $\operatorname{ob} \mathcal{C}$ of *objects*,
- A class $\operatorname{mor} \mathcal{C}$ of *morphisms*,
- A function $\operatorname{dom} : \operatorname{mor} \mathcal{C} \to \operatorname{ob} \mathcal{C}$,
- A function $\operatorname{cod} : \operatorname{mor} \mathcal{C} \to \operatorname{ob} \mathcal{C}$,
- A function $\operatorname{id} : \operatorname{ob} \mathcal{C} \to \operatorname{mor} \mathcal{C}$,
- A function $\circ : \{(g, f) \in \operatorname{mor} \mathcal{C} \times \operatorname{mor} \mathcal{C} : \operatorname{dom} g = \operatorname{cod} f\} \to \operatorname{mor} \mathcal{C}$;

satisfying the following axioms:

1. For all $x \in \operatorname{ob} \mathcal{C}$, we have $\operatorname{dom} \operatorname{id} x = x = \operatorname{cod} \operatorname{id} x$,
2. For all $f \in \operatorname{mor} \mathcal{C}$, we have $f \circ \operatorname{id} \operatorname{dom} f = f = \operatorname{id} \operatorname{cod} f \circ f$,
3. $\circ$ is associative, i.e. $f \circ (g \circ h) = (f \circ g) \circ h$ whenever this makes sense.

In this description, we get for any two objects $x, y \in \operatorname{ob} \mathcal{C}$ a *hom-class*

$$\operatorname{Hom}_{\mathcal{C}}(x, y) := \{f \in \operatorname{mor} \mathcal{C} : \operatorname{dom} f = x \wedge \operatorname{cod} f = y\}.$$

If $\operatorname{Hom}_{\mathcal{C}}(x, y)$ is a set for all $x, y \in \operatorname{ob} \mathcal{C}$, we say that $\mathcal{C}$ is *locally small.* If $\operatorname{mor} \mathcal{C}$ is a set, we say that $\mathcal{C}$ is *small.* If $\mathcal{C}$ is not small we say that it is *large.*

It is important to note that small categories are necessarily locally small, but the converse does not hold in general.

Proposition A.1.8. *Let $\mathcal{C}$ be a category. The following are equivalent:*

1. *$\mathcal{C}$ is small,*
2. *$\mathcal{C}$ is locally small and $\operatorname{ob}\mathcal{C}$ is a set.*

Proof. First, suppose $\mathcal{C}$ is small, i.e. $\operatorname{mor}\mathcal{C}$ is a set. Then for all $x, y \in \operatorname{ob}\mathcal{C}$ we have

$$\operatorname{Hom}_{\mathcal{C}}(x, y) \subseteq \operatorname{mor}\mathcal{C}$$

and since $\operatorname{mor}\mathcal{C}$ is a set it follows that $\operatorname{Hom}_{\mathcal{C}}(x, y)$ is a set[a]. Thus, $\mathcal{C}$ is locally small.

Next, since $\operatorname{mor}\mathcal{C}$ is a set, we have that

$$I := \{f \in \operatorname{mor}\mathcal{C} : \exists x \in \operatorname{ob}\mathcal{C}\,(f = \operatorname{id} x)\} \subseteq \operatorname{mor}\mathcal{C}$$

is a set[b]. Now

$$\operatorname{ob}\mathcal{C} = \{\operatorname{dom} f : f \in I\}$$

and $\{\operatorname{dom} f : f \in I\}$ is a set[c].

In the other direction, suppose $\mathcal{C}$ is locally small and that $\operatorname{ob}\mathcal{C}$ is a set. Now[d] we obtain the set

$$\bigcup_{x \in \operatorname{ob}\mathcal{C}} \bigcup_{y \in \operatorname{ob}\mathcal{C}} \operatorname{Hom}_{\mathcal{C}}(x, y),$$

which equals $\operatorname{mor}\mathcal{C}$. □

[a] exercise!
[b] by the same exercise!
[c] by ZFC's *axiom schema of replacement*
[d] by the *union* axiom of ZFC, applied to the family produced by the axiom schema of replacement

This allows us to produce the category Set of all sets, the category Grp of all groups, etc. We may also produce the category Cat of all *small* categories! That's because the data of a small category consists of sets and functions between them, so there is a class of all small categories. On the other hand, we may *not* produce the category of all categories, since the data of an arbitrary large category consists of proper classes and functions between them, and so there is no class of large categories (the elements of a class must be sets!).

A.2 Universes

Classes are somewhat elegant and useful, but also a bit troublesome: sometimes we *would* like to have access to the category of all categories! For example, it would be convenient to have a functor which sends a category $\mathcal{C}$ to its presheaf category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set})$. Even if $\mathcal{C}$ is a small category, its presheaf category can be large (e.g. take $\mathcal{C}$ to be the terminal category)! The formalism of classes allows us to construct large collections of sets, but it doesn't allow us to construct (even small) collections of classes. To obviate this difficulty, Grothendieck introduced the idea of *universes*, which allow us to filter the class of all sets into layers which we might call "small sets", "large sets", "huge sets", etc. Each of these layers will be a set, i.e. there will be a set of all small sets, and a set of all large sets, and a set of all huge sets, etc!

To be more precise: Grothendieck defined a notion of *universe*, which is just a condition any given set may or may not satisfy. If a set U is a universe, then we should think about U as "the set of all U-small sets". If we have two universes U_1 and U_2 with $U_1 \subsetneq U_2$, then we might (for convenience) call the elements of U_1 "small sets" and the elements of U_2 "large sets". If we have three universes, ... you get the idea.

Then if we have a universe U_1, we can define a "U_1-small category" to be a category whose class of morphisms is an element of U_1 (and in particular is a set). If $U_1 \subsetneq U_2$, it will turn out that the class of all U_1-small categories is an element of $U_2 - U_1$, and we will be able to define the category of all U_1-small categories, which will itself be U_2-small (but not U_1-small). And so on, and so forth. Importantly, no matter how many nested universes we have, each universe is a *set*, and so we can deal with its elements and subsets and so on with ordinary set-theory, without ever mentioning classes!

The final philosophical point to make, before we actually define universes, is that at the end of the day, it will always suffice to prove our theorems in some sufficiently large universe – it is never actually necessary to consider the entire class of sets all at once. After all, any particular mathematical object has a fixed size. So, the plan for setting up category theory with Grothendieck universes is to *only* use small categories (as defined in the previous subsection), but to treat certain sets as "small" (these small sets will include e.g. all topological spaces of interest to us in some mathematical context) and other sets as "large" (these large sets will include e.g. the set of all small sets).

Definition A.2.1. A set U is said to be a *(Grothendieck) universe* if

1. U is transitive, i.e.
$$\forall x \forall y (x \in y \in U \implies x \in U),$$
2. U is closed under forming pairs, i.e.
$$\forall x, y \in U(\{x, y\} \in U),$$
3. U is closed under forming power sets, i.e.
$$\forall x \in U(\mathcal{P}(x) \in U),$$
where $\mathcal{P}(x)$ denotes the power set of x,

4. U is closed under unions, i.e.

$$\forall I \in U \, \forall f : I \to U \left(\bigcup_{i \in I} f(i) \in U \right),$$

5. $\omega \in U$, where $\omega = \{0, 1, 2, \dots\}$ is the set of finite (von Neumann) ordinals.

There are other equivalent ways to express these axioms, but the point is that when U is a universe in ZFC set theory, $(U, \in)$ is a model of ZFC set theory. Some authors (including Grothendieck) do not require the last axiom (in which case $(U, \in)$ will be a model of ZFC minus the axiom of infinity). Another way to say this is that for any universe U, we can perform any set-theoretic construction with elements of U and remain inside U.

Definition A.2.2. Let U be a universe. By a "U-small set" we will mean simply an element of U. By a "U-small category" we will mean simply a small category $\mathcal{C}$ such that $\operatorname{ob}\mathcal{C} \in U$ and $\operatorname{mor}\mathcal{C} \in U$.

We note that there is a set of all U-small sets (namely, U), and there is a set of all U-small categories, but of course neither the set of all U-small sets nor the set of all U-small categories will be U-small.

It is worth pointing out that universes are quite large: every set you've ever seen is U-small for *every* universe U. That's because every set you've ever explicitly constructed has been built from some iterated application of basic set theory constructions, beginning with the empty set and the set ω, and U is closed under all such constructions[6]! Basic set theory considerations tell us that there are sets much larger than any set you've ever explicitly constructed, and Russell's paradox tells us that we cannot hope to deal with the enormity of the collection of absolutely all sets, but the point of universes is that we don't have to: we can instead restrict our attention to a universe U, which will contain every set we care about, and simply work within that universe.

There is only one problem: it is impossible to prove that universes exist[7]. Instead, we add a new axiom to our set theory which imposes their existence.

Axiom A.2.3 (Axiom of Universes)**.** For all sets x, there exists a universe U such that $x \in U$.

It is possible in principle that the universe axiom is inconsistent with the other axioms of ZFC, but this is widely believed not to be the case. This belief is not weakly held; set theorists have actually spent a lot of time thinking about this axiom, in the following guise:

Proposition A.2.4. *In ZFC, the following two statements are equivalent:*

1. The universe axiom holds,

[6] The only missing step here is to show that $\varnothing \in U$ for all universes U: exercise!

[7] Unless ZFC is inconsistent, of course. More precisely, ZFC plus "there exists at least one universe" proves that ZFC is consistent.

2. *For all cardinals κ, there exists a strongly inaccessible*[8] *cardinal λ such that $\lambda > \kappa$.*

Statement (2) above is an example of what set theorists call a *large cardinal axiom*, which are an important topic in modern set theory. In fact, it turns out that universes are in bijective correspondence with strongly inaccessible cardinals – for every strongly inaccessible cardinal λ, the von Neumann stage V_λ is a universe, and every universe is equal to V_λ for some strongly inaccessible cardinal λ[9]. To summarize: **the best experts believe strongly that the universe axiom is not inconsistent with the other axioms of ZFC**, but it is impossible to prove this using the axioms of ZFC unless ZFC is itself inconsistent!

Some mathematicians tend to feel uncomfortable with this situation. This fear comes from a good place, but if you find yourself unsettled by this, please note that one could raise the exact same objection about (e.g.) the power set axiom, which simply asserts that every set has a power set. Just as we should be happy to assume (axiomatically) that power sets exist, we should be happy to assume (axiomatically) that universes exist.

In practice, you will often find technical setups of the following form:

Convention A.2.5. Fix universes U_1 and U_2 with $U_1 \in U_2$[10]. Let Set denote the category of U_1-small sets, and let SET denote the category of U_2-small sets. Let Cat denote the category of U_1-small categories, and let CAT denote the category of U_2-small categories.

Then we have (for example) that Set is an object of CAT (exercise!), and we operate *as if* Set is the category of all sets and CAT is the category of all categories, even though this is not literally true. Indeed, all of the categories Set, SET, Cat, CAT described in Convention A.2.5 are small in the sense that the class of morphisms in each is a set.

So, our notation here conflicts with our earlier choice to use Set to denote the category of *all* sets. Besides this kind of notational conflict, the theory of classes does not interfere with the theory of universes in any way – i.e. it is perfectly fine to discuss proper classes in a setting where one has assumed the universe axiom.

We note now that this setup allows us to define the free cocompletion functor

$$Y : \mathsf{Cat} \to \mathsf{CAT},$$

which sends a U_1-small category $\mathcal{C} \in \operatorname{ob} \mathsf{Cat}$ to its category of presheaves $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set})$, and which sends a functor $f : \mathcal{C} \to \mathcal{D}$ to left Kan extension along f^{op}. Note that $Y(\mathcal{C})$ is U_2-small but typically not U_1-small, so Y really does need CAT (rather than Cat) as its target. We should also be clear that

[8] A cardinal μ is said to be *strongly inaccessible* if it is uncountable, every unbounded subset of μ has cardinality μ, and $2^\nu < \mu$ for every cardinal $\nu < \mu$.

[9] Precisely, for λ equal to the cardinality of the universe.

[10] The universe axiom says in particular that we may make such a choice. It follows from $U_1 \in U_2$ that $U_1 \subsetneq U_2$, and in fact $U_1 \subsetneq U_2$ is equivalent to $U_1 \in U_2$ for any universes U_1, U_2.

$Y(\mathcal{C})$ is the free U_1-*cocompletion* of $\mathcal{C}$, i.e. $Y(\mathcal{C})$ admits all U_1-small colimits, and all objects of $Y(\mathcal{C})$ are U_1-small colimits of representable presheaves. Since Cat is a full subcategory of CAT, we may now compare Y with the inclusion $i : \mathsf{Cat} \hookrightarrow \mathsf{CAT}$, and thereby express the Yoneda embedding as a single natural transformation – a statement which we could not even write down using classes alone.

Proposition A.2.6. *Let $i : \mathsf{Cat} \hookrightarrow \mathsf{CAT}$ denote the inclusion of the full subcategory of U_1-small categories. The Yoneda embeddings*[11]

$$y_{\mathcal{C}} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}) = Y(\mathcal{C})$$

are the components of a natural transformation $y : i \Rightarrow Y$.

The Yoneda lemma itself (the family of bijections $\mathrm{Hom}_{Y(\mathcal{C})}(y_{\mathcal{C}}(x), F) \cong F(x)$) can be packaged in the same spirit, as a natural isomorphism between two functors defined on a suitable category of triples $(\mathcal{C}, x, F)$ with $\mathcal{C} \in \mathrm{ob}\,\mathsf{Cat}$, $x \in \mathrm{ob}\,\mathcal{C}$, and $F \in \mathrm{ob}\,Y(\mathcal{C})$.

Slogan 3. Size issues are unavoidable. However you choose to deal with them, you must always pay attention to which things are large and which things are small.

[11] One should of course carefully define the Yoneda embedding and check that it is well-defined: since $\mathcal{C}$ is U_1-small, each hom-set $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is indeed an object of Set.