A Linear Map with no Adjoint

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Let $d : C^\infty[0,1] \to C^\infty[0,1]$ be the derivative operator, where $C^\infty[0,1]$ is the real vector space of infinitely differentiable functions $[0,1] \to \mathbb{R}$. Suppose for contradiction that d has an adjoint $d^* : C^\infty[0,1] \to C^\infty[0,1]$ with respect to the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x$.

Let $Q = d + d^*$. For all $f, g \in C^\infty[0,1]$, we have

$$\begin{aligned}\langle Q(f), g\rangle &= \langle d(f), g\rangle + \langle d^*(f), g\rangle = \langle d(f), g\rangle + \langle f, d(g)\rangle \\ &= \langle f', g\rangle + \langle f, g'\rangle = \int_0^1 f'(x)g(x)\,\mathrm{d}x + \int_0^1 f(x)g'(x)\,\mathrm{d}x \\ &= \int_0^1 (f'g + fg')(x)\,\mathrm{d}(x) = \int_0^1 (fg)'(x)\,\mathrm{d}(x) = f(1)g(1) - f(0)g(0).\end{aligned}$$

In particular, $\langle Q(x), x^n\rangle = 1$ for all natural numbers n. By the Cauchy-Schwarz inequality, we have

$$1 = |\langle Q(x), x^n\rangle| \leq \|Q(x)\|\|x^n\| = \frac{\|Q(x)\|}{\sqrt{2n+1}}$$

for all natural numbers n. Therefore,

$$0 = \|Q(x)\| \lim_{n\to\infty} \frac{1}{\sqrt{2n+1}} = \lim_{n\to\infty} \frac{\|Q(x)\|}{\sqrt{2n+1}} \geq \lim_{n\to\infty} 1 = 1,$$

which is a contradiction. We conclude that d does not have an adjoint.