Classification of Finite-Dimensional Composition-Closed Vector Spaces of Functions

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1 Introduction

The following was originally written as a solution to a problem posted on math.stackexchange.com [1]. The post asked if finite-dimensional vector subspaces of $C^\infty(\mathbb{R})$ which are closed under composition could be easily classified. The answer turns out to be yes, and indeed the classification goes through for all vector spaces of differentiable functions (with no assumption on smoothness).

2 Solution

Definition 1. $D(\mathbb{R})$ is the (real) vector space of differentiable functions $\mathbb{R} \to \mathbb{R}$, under pointwise addition and scalar multiplication.

Note: $D(\mathbb{R})$ is somewhat larger than the more common space $C^1(\mathbb{R})$ of *continuously* differentiable functions.

Definition 2. A *composition space* is a finite-dimensional $\mathbb{R}$-subspace of $D(\mathbb{R})$ which is closed under composition. A composition space is said to be *affine* if it consists only of affine-linear functions (that is, functions of the form $x \mapsto ax + b$ for some $a, b \in \mathbb{R}$), and *non-affine* otherwise.

Lemma 1. *All* 1*-dimensional composition spaces are affine.*

Proof. Let V be a 1-dimensional composition space. Let $f \in V \setminus \{0\}$ so that f generates V. This means there is some function $\lambda : \mathbb{R} \to \mathbb{R}$ such that $f \circ \alpha f = \lambda(\alpha) f$ for all $\alpha \in \mathbb{R}$. Since $f \neq 0$, $X := f^{-1}(\mathbb{R} \setminus \{0\})$ is a nonempty open set, whence $f(X) = f(\mathbb{R}) \setminus \{0\}$ has nonempty interior. Now, choose some $y \in f(X)$, so that $y = f(x_0)$ for some $x_0 \in X$. We see that $f(\alpha y) = f(\alpha f(x_0)) = \lambda(\alpha) f(x_0) = \lambda(\alpha) y$, so $\lambda(\alpha) = f(\alpha y)/y$. Since f is differentiable, this shows that λ is differentiable. Indeed, $\lambda'(\alpha) = f'(\alpha y)$. Since the choice of y was arbitrary and the left-hand side does not depend on y, f' must be constant on the set $\alpha f(X) = \{\alpha y : y \in f(X)\}$ for all $\alpha \in \mathbb{R}$. But the sets $\alpha f(X)$ cover $\mathbb{R}$, so f' is constant. Thus, f is affine linear, whence V is affine. $\square$

Theorem 1. *All composition spaces are affine. In more detail, the only composition spaces are:*

- 0
- $\{x \mapsto a : a \in \mathbb{R}\}$
- $\{x \mapsto ax : a \in \mathbb{R}\}$
- $\{x \mapsto ax + b : a, b \in \mathbb{R}\}$

Proof. Suppose for contradiction that there exists a non-affine composition space. Let V be a non-affine composition space of minimal dimension. The above lemma tells us that $\dim V \geq 2$. Now suppose for contradiction that there is some $F \in V$ for which $F(0) \neq 0$. Then the linear map $e := f \mapsto f(0) : V \to \mathbb{R}$ is surjective, so $\ker e$ is a subspace of V of codimension 1. Since $\ker e$ is closed under composition, minimality of $\dim V$ tells us that $\ker e$ is affine. V contains the nonzero constant function $F \circ 0 = x \mapsto F(0)$, which is not in $\ker e$, so V is generated by $\ker e \cup \{F \circ 0\}$. But this means that V is affine; a contradiction. Thus, $f(0) = 0$ for all $f \in V$.

Next, let $F, G \in V \setminus \{0\}$ be arbitrary. Since $F \neq 0$, there is some $x_0 \in \mathbb{R}$ such that $F(x_0) \neq 0$. Let $F_1 = \frac{x_0}{F(x_0)} F$ and define $T := f \mapsto f \circ F_1 : V \to V$. T is linear and nontrivial since $T(F)(x_0) = F(F_1(x_0)) = F(x_0) \neq 0$. Thus, $\ker T$ has positive codimension in V. Since $\ker T$ is also closed under composition, $\ker T$ is affine. Since $f(0) = 0$ for all $f \in V$, every element of $\ker T$ is linear. Now if $x \mapsto ax \in \ker T$, we have that $aF_1 = (x \mapsto ax) \circ F_1 = 0$, so $a = 0$. This shows that $\ker T = 0$, so T is injective. Since V is finite-dimensional, T is surjective. Thus, $G = H \circ F_1$ for some $H \in V$. Now, whenever $F'(x) = 0$, we have $F_1'(x) = 0$, so $G'(x) = (H \circ F_1)'(x) = H'(F_1(x))F_1'(x) = 0$. In other words, $(F')^{-1}(0) \subseteq (G')^{-1}(0)$. Symmetrically, $(G')^{-1}(0) \subseteq (F')^{-1}(0)$. F and G were arbitrary, so $(F')^{-1}(0) = (G')^{-1}(0)$ for all nonzero $F, G \in V$.

Now, since $\dim V \geq 2$, let $f, g \in V$ be linearly independent. Since $f \neq 0$ and $f(0) = 0$, f is non-constant, so there is some $s \in \mathbb{R}$ such that $f'(s) \neq 0$. Since $g \neq 0$, $g'(s) \neq 0$. Now $h := f + \frac{-f'(s)}{g'(s)} g$ is a nontrivial linear combination of f and g, hence $h \neq 0$, so $h'(s) \neq 0$. But $h'(s) = f'(s) + \frac{-f'(s)}{g'(s)} g'(s) = 0$, giving a contradiction.

We conclude that all composition spaces are affine, and we must only complete the classification of affine composition spaces. The space of all affine-linear functions is 2-dimensional, so any nontrivial proper subspace is 1-dimensional. If $f = x \mapsto ax + b$ generates a composition space, then $f \circ 0 = x \mapsto b$ is an element of the composition space. This means that $b = 0$ (in which case the composition space is $\{x \mapsto ax : a \in \mathbb{R}\}$) or $b \neq 0$ (in which case the composition space is $\{x \mapsto a : a \in \mathbb{R}\}$). □

References

[1] eepperly16. *Finite dimensional real function spaces closed under composition.* Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/3358169.