Hall Matching via Krein-Milman

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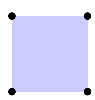
Thanks to Clark Lyons for teaching me that this argument is possible :)

A Recap of the Krein-Milman Theorem

We will discuss only the finite-dimensional case of Krein-Milman, since this is all we need for the proof of Hall's matching theorem. Let V be a real vector space. A subset $K \subseteq V$ is said to be *convex* if, for all $x, y \in C$ and all $t \in [0, 1]$, we have $tx + (1-t)y \in K$. An element $e \in K$ is said to be an *extreme point* if, for all $x, y \in K$ and $t \in (0, 1)$, $tx + (1-t)y = e$ implies $x = y = e$. In other words, an extreme point is a point in K which does *not* lie strictly between two other points. Here is an example of a convex subset of $\mathbb{R}^2$, with its extreme points highlighted:



Finally, given any subset $S \subseteq V$, its *convex hull* $\text{hull}(S)$ is the smallest convex subset of V which contains S. $\text{hull}(S)$ is well-defined because the intersection of any collection of convex subsets of V is convex, and V is a convex subset of V which contains S – thus, we may construct $\text{hull}(S)$ as the intersection of all convex subsets of V which contain S.

Theorem (Krein-Milman)**.** *Let C be a compact convex subset of $\mathbb{R}^n$. Then C is equal to the convex hull of its set of extreme points.*

It is worth noting that this theorem applies just as well to subsets of any finite-dimensional real vector space V, by simply choosing any isomorphism $V \to \mathbb{R}^n$. In this case, "compact" means "has compact image in $\mathbb{R}^n$", and the choice of isomorphism does not matter because every vector space automorphism of $\mathbb{R}^n$ preserves compact subsets.

Corollary. *Let C be a nonempty compact convex subset of $\mathbb{R}^n$. Then C has at least one extreme point.*

Proof. The convex hull of $\varnothing$ is $\varnothing$. □

Hall's Matching Theorem

Let G be a finite d-regular bipartite graph, with bipartition $\{X, Y\}$. Let E denote the set of edges in G (which we encode as cardinality-2 subsets of G), and let $\sim$ denote the adjacency relation on G. For any $S \subseteq G$, we let

$$\partial S := \{y \in G : \exists x \in S (x \sim y)\}$$

denote the set of vertices of G which are adjacent to elements of S. Note that when $S \subseteq X$ we have $\partial S \subseteq Y$.

Theorem (Hall)**.** *If $d > 0$ and $|\partial S| \geq |S|$ for all $S \subseteq X$, then there exists a perfect matching on G, i.e. a subset of $\sim$ which is a bijection $X \to Y$.*

Proof. Let $V := \mathbb{R}^E$ denote the $\mathbb{R}$-vector space of functions $E \to \mathbb{R}$, which we note is finite-dimensional. For each $f \in V$, we let $\overline{f} : G \to \mathbb{R}$ denote the function

$$\overline{f}(x) = \sum_{\substack{y \in G \\ x \sim y}} f(\{x, y\}).$$

Define

$$K := \{f \in [0,1]^E : \overline{f} = 1\}$$

where 1 denotes the constant function. We note that K is a compact convex subset of V, and we see that K is nonempty because the constant function $1/d$ is an element of K (by the d-regularity of G). By the Krein-Milman theorem, there exists an extreme point $f \in K$.

Let $E' = \{e \in E : f(e) \notin \{0, 1\}\}$, and suppose for contradiction that $E' \neq \varnothing$. Let H be the subgraph of G spanned by E', and note that $\deg(h) \geq 2$ for all $h \in H$. Thus, H contains some cycle C. Because G is bipartite, C has even length. Pick a function $s : C \to \{-1, 1\}$ such that $s(e) = -s(e')$ whenever e and e' share a vertex, i.e. s is an alternating choice of sign for each edge in the cycle C.

Now let $\varepsilon = 1/2 - \max\{|f(e) - 1/2| : e \in E'\}$, so that $f(e) + \varepsilon \leq 1$ and $f(e) - \varepsilon \geq 0$ for all $e \in E'$.

Finally, let $g_+, g_- : E \to \mathbb{R}$ be the functions

$$g_\pm = f \pm (\varepsilon/2)s.$$

We note that $g_+, g_- \in K$, and $\frac{1}{2}g_+ + \frac{1}{2}g_- = f$. We conclude that $g_+ = g_- = f$, so $\varepsilon = 0$, and thus $f(e) \in \{0, 1\}$ for some $e \in E$, contradicting the definition of E.

We conclude that $E = \varnothing$, and so f is integer-valued. Now

$$\{(x, y) : f(\{x, y\}) = 1\}$$

is a bijection $X \to Y$. □