Presentations of Burnside Rings

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July 2025

1 Introduction

Let G be a finite group. The *Burnside ring* $A(G)$ of G is the Grothendieck ring of the category of finite G-sets (with respect to $\amalg$ and $\times$). In other words, $A(G)$ is generated as an abelian group by isomorphism classes of finite G-sets, subject to the relations $[X \amalg Y] = [X] + [Y]$, and multiplication is given by $[X][Y] = [X] \times [Y]$ (extended linearly). For any subgroup $H \leq G$, the forgetful functor from G-sets to H-sets admits both a left adjoint (called *induction*) and a right adjoint (called *coinduction*). These three functors then give rise to operations $\mathrm{tr} : A(H) \to A(G)$ (called *transfer*, induced by induction), $\mathrm{nm} : A(H) \to A(G)$ (called *norm*, induced by coinduction), and $\mathrm{res} : A(G) \to A(H)$ (called *restriction*, induced by the forgetful functor). These are the aforementioned "TNR operations", also sometimes called "Tambara operations". The most important things to know about these operations are:

- res is a ring homomorphism (because the forgetful functor preserves both products and coproducts);
- tr is a homomorphism of additive groups (because induction preserves coproducts);
- nm is a homomorphism of multiplicative monoids (because coinduction preserves products).

In this document, I want to record the basic principles for computing (presentations of) Burnside rings and their TNR operations, a topic I feel is under-discussed in the literature. There will be many details I gloss over or do not discuss here – the intent is simply to equip the reader with the tools to compute these things when needed.

2 The Table of Marks

Since every finite G-set decomposes uniquely as a disjoint union of orbits, the Burnside ring $A(G)$ is generated freely (as an abelian group) by the classes $[G/H]$ where H ranges over the conjugacy classes of subgroups of G. For each subgroup $K \leq G$, we also get a ring homomorphism called the *K-mark homomorphism* $\varphi_K : A(G) \to \mathbb{Z}$, determined by $[X] \mapsto \#X^H$. The mark homomorphisms for conjugate subgroups are equal. The most important fact about Burnside rings is that the map

$$(\varphi_K)_{[K]} : A(G) \to \prod_{[K]} \mathbb{Z}$$

is injective (where $[K]$ ranges over conjugacy classes of subgroups of G). Thus, one can read off a presentation for the commutative ring $A(G)$ by computing the *table of marks*, which is the transpose of[1] the square matrix representing this injection with respect to the basis $\{[G/H] : [H]\}$ for $A(G)$. That is, the columns of the table of marks are indexed by a chosen ordered set of representatives[2] $(H_1, \ldots, H_k)$ of the conjugacy classes of G, and the rows are then indexed by the integral basis $([G/H_1], \ldots, [G/H_k])$ for $A(G)$. The (i, j)-entry of the table of marks is thus $\#(G/H_i)^{H_j}$.

You can compute a table of marks with `GAP` via `Display(TableOfMarks(G));` where `G` is the group in question.

[1] Why the transpose?? I use this convention because `GAP` does.

[2] By convention, the total order is always chosen to refine the order of the subgroups. This guarantees that the table of marks is lower-triangular and the first column is non-increasing.

Example 1. Let $G = C_p$. The table of marks is

$\#$	e	C_p
C_p/e	p	0
C_p/C_p	1	1

The class $[C_p/C_p]$ is the unit in $A(C_p)$, of course. The class $[C_p/e]$ is traditionally denoted t, and we see from the table of marks that $t^2 = pt$ holds in $A(C_p)$. We conclude that

$$A(C_p) \cong \mathbb{Z}[t]/(t^2 - pt).$$

Example 2. Let $G = C_{p^2}$. The table of marks is

$\#$	e	C_p	C_{p^2}
C_{p^2}/e	p^2	0	0
C_{p^2}/C_p	p	p	0
C_{p^2}/C_{p^2}	1	1	1

Let's set $t = [C_{p^2}/C_p]$ and $u = [C_{p^2}/e]$. We see that $u^2 = p^2u$, $t^2 = pt$, and $tu = pu$. We conclude that

$$A(C_{p^2}) \cong \mathbb{Z}[t, u]/(u^2 - p^2u, t^2 - pt, tu - pu).$$

Example 3. Let $G = S_3$. The table of marks is

$\#$	e	C_2	C_3	S_3
S_3/e	6	0	0	0
S_3/C_2	3	1	0	0
S_3/C_3	2	0	2	0
S_3/S_3	1	1	1	1

Let's set $x_6 = [S_3/e]$, $x_3 = [S_3/C_2]$, and $x_2 = [S_3/C_3]$ (the subscripts correspond to the cardinalities of the S_3-sets). We get

$$\begin{aligned} x_6^2 &= 6x_6 \\ x_3^2 &= x_6 + x_3 \\ x_2^2 &= 2x_2 \\ x_6x_3 &= 3x_6 \\ x_6x_2 &= 2x_6 \\ x_3x_2 &= x_6 \end{aligned}$$

The last equation allows us to eliminate x_6 as a generator, and the relations become

$$\begin{aligned} x_3^2x_2^2 &= 6x_3x_2 \\ x_3^2 &= x_3x_2 + x_3 \\ x_2^2 &= 2x_2 \\ x_3^2x_2 &= 3x_3x_2 \\ x_3x_2^2 &= 2x_3x_2 \end{aligned}$$

Now note that the second and third equations make the the rest redundant! So we have simply

$$A(S_3) \cong \mathbb{Z}[x_3, x_2]/(x_3^2 - x_3x_2 - x_3, x_2^2 - 2x_2).$$

Example 4. Let $G = C_{pq}$. The table of marks is

$\#$	e	C_p	C_q	C_{pq}
C_{pq}/e	pq	0	0	0
C_{pq}/C_p	q	q	0	0
C_{pq}/C_q	p	0	p	0
C_{pq}/C_{pq}	1	1	1	1

We see immediately that $[C_{pq}/e] = [C_{pq}/C_p][C_{pq}/C_q]$. So, $A(C_{pq})$ is generated by $x_q := [C_{pq}/C_p]$ and $x_p := [C_{pq}/C_q]$. The relations are

$$x_q^2 = qx_q$$
$$x_p^2 = px_p$$

and so we get

$$A(C_{pq}) \cong \mathbb{Z}[x_q, x_p](x_q^2 - qx_q, x_p^2 - px_p).$$

It may seem suspicious that $A(C_{pq}) \cong A(C_p) \otimes A(C_q)$, and it is!

Proposition 1. *Let G_1 and G_2 be finite groups of relatively prime order. Then $A(G_1 \times G_2) \cong A(G_1) \otimes A(G_2)$.*

Proof. This follows from the fact that every subgroup of $G_1 \times G_2$ is of the form $H_1 \times H_2$. □

Example 5. Let $G = C_p \times C_p$. It helps to think of G as the vector space $\mathbb{F}_p^2$, so that its (conjugacy classes of) subgroups are just its subspaces. Then we easily see that there are $p + 3$ conjugacy classes of subgroups (the trivial subspace, $p + 1$ lines, and the whole space). Let $L_0, \dots, L_p$ denote the lines. To compute the table of marks, we use the following observation: when K is not subconjugate to H, $\#(G/H)^K = 0$. Then we see easily that the table of marks looks like:

$\#$	e	L_0	L_1	$\dots$	L_p	G
G/e	p^2	0	0	$\dots$	0	0
G/L_0	p	p	0	$\dots$	0	0
G/L_1	p	0	p	$\dots$	0	0
$\vdots$				$\ddots$		
G/L_p	p	0	0	$\dots$	p	0
G/G	1	1	1	$\dots$	1	1

and thus we have

$$A(C_p \times C_p) \cong \mathbb{Z}[v, \ell_0, \dots, \ell_p]/(\ell_i^2 - p\ell_i, \ell_i\ell_j - v : i \neq j).$$

The generator v is redundant but makes the presentation easier to read.

3 Computing Restriction, Transfer, and Norm Maps

Let G be a finite group, and let H be a subgroup of G. The functions

$$\mathrm{res}_H^G : A(G) \to A(H)$$
$$\mathrm{tr}_H^G : A(H) \to A(G)$$
$$\mathrm{nm}_H^G : A(H) \to A(G)$$

all have straightforward descriptions – they are induced by canonical functors between the categories $G-\mathsf{set}$ and $H-\mathsf{set}$.

3.1 Restriction

res_H^G is a ring homomorphism induced by the forgetful functor $G-\mathsf{set} \to H-\mathsf{set}$. In particular, this means that

$$\varphi_K \,\mathrm{res}_H^G[G/L] = \varphi_K[G/L]$$

for all subgroups $K \leq H$. This makes res_H^G very easy to compute!

Example 6. Let's compute the restriction $A(S_3) \to A(C_3)$. Recall that

$$A(S_3) \cong \mathbb{Z}[x_3, x_2]/(x_3^2 - x_3x_2 - x_3, x_2^2 - 2x_2)$$

and

$$A(C_3) \cong \mathbb{Z}[t]/(t^2 - 3t),$$

Here are the tables of marks, with rows labelled according to our presentations for the Burnside rings, and columns highlighted to indicate the subgroup inclusion $C_3 \to S_3$.

	e	C_3
t	3	0
	1	1

	e	C_2	C_3	S_3
	6	0	0	0
x_3	3	1	0	0
x_2	2	0	2	0
	1	1	1	1

We see from this that $\mathrm{res}^{S_3}_{C_3}(x_3)$ has mark $(3, 0)$, and thus $\mathrm{res}^{S_3}_{C_3}(x_3) = t$. Likewise, $\mathrm{res}^{S_3}_{C_3}(x_2) = 2$. Since $\mathrm{res}^{S_3}_{C_3}$ is a ring homomorphism, this is all we need to know!

3.2 Transfer

tr^G_H is an abelian group homomorphism induced by the induction functor $H\text{−}\mathsf{set} \to G\text{−}\mathsf{set}$. Induction sends H/K to G/K, which makes things very easy. On the other hand, tr^G_H is usually not a ring homomorphism. However, $\mathrm{tr}^G_H(\mathrm{res}^G_H(a)b) = a\,\mathrm{tr}^G_H(b)$ (i.e. tr^G_H is $A(G)$-linear), which often helps.

Example 7. Let's compute the transfer $A(C_3) \to A(S_3)$. Here are the tables of marks again, now with the rows highlighted to indicate the subgroup inclusion $C_3 \to S_3$.

$[C_3/e] = \mathrm{res}(x_3) = t$	3	0
$[C_3/C_3] = \mathrm{res}(1) = 1$	1	1

$[S_3/e]$	6	0	0	0
$[S_3/C_2] = x_3$	3	1	0	0
$[S_3/C_3] = x_2$	2	0	2	0
$[S_3/S_3]$	1	1	1	1

So, for $a, b \in \mathbb{Z}$, we get $\mathrm{tr}^{S_3}_{C_3}(a + bt) = ax_2 + bx_3x_2$.

3.2.1 Norm

nm^G_H is a homomorphism of multiplicative monoids[3] induced by the coinduction functor $H\text{−}\mathsf{set} \to G\text{−}\mathsf{set}$. Coinduction sends an H-set X to $\mathrm{Map}_H(G, X)$, where we view G as an H set via left multiplication, and G acts on this set of functions by precomposition with right multiplication.

Since nm^G_H is not additive and the multiplicative monoid of $A(H)$ is never finitely generated, it tends to be annoying to find explicit formulae for nm^G_H. However, there is one nice thing we can spot directly:

$$\mathrm{Map}_H(G, X)^K = \{f : G \to X \mid \forall h \in H \forall g \in G \forall k \in K(f(hgk) = hf(g))\}$$

In other words, $\mathrm{Map}_H(G, X)^K$ is the set of H-equivariant maps $G \to X$ which are constant on left K-cosets. For example, we have $\#\mathrm{Map}_H(G, X)^G = \#X^H$. It's often possible to use this to compute the mark of $\mathrm{nm}^G_H([X])$.

To extend to arbitrary elements of $A(H)$ (i.e. virtual isomorphism classes of H-sets), we need to use either a Mazur sum formula [1] or Tambara's original construction.

Other useful tools I know of:

1. Thinking really hard;
2. The double-coset formula for $\mathrm{res}^G_H \circ \mathrm{nm}^G_H$.

[3] also satisfying $\mathrm{nm}(0) = 0$

Example 8. Let's compute the norm $A(C_3) \to A(S_3)$. The good news is that a Mazur sum formula is not too bad here, because $[S_3 : C_3] = 2$. We have $\mathrm{nm}(a + b) = \mathrm{nm}(a) + \mathrm{nm}(b) + \mathrm{tr}(a\bar{b})$, where $\bar{b}$ denotes the (unique) nontrivial Weyl conjugate of b. Here the Weyl action on $A(C_3)$ is trivial, because every subgroup of C_3 is normal in S_3. Thus, we have

$$\mathrm{nm}(a + b) = \mathrm{nm}(a) + \mathrm{nm}(b) + \mathrm{tr}(ab).$$

Next, we compute $\mathrm{nm}(k)$ for k a natural number. This is the class of the S_3-set $\mathrm{Map}_{C_3}(S_3, (C_3/C_3)^{\amalg k})$. Any C_3-equivariant map $S_3 \to (C_3/C_3)^{\amalg k}$ is constant on left C_3-cosets, so is fixed by the action of C_3 on $\mathrm{Map}_{C_3}(S_3, (C_3/C_3)^{\amalg k})$. Thus, $\mathrm{Map}_{C_3}(S_3, (C_3/C_3)^{\amalg k})$ has only orbits of types S_3/C_3 and S_3/S_3. An orbit of type S_3/S_3 corresponds to a constant function, and there are k of these. There are k^2-many C_3-equivariant functions in total, so

$$\mathrm{nm}(k) = k + \frac{k^2 - k}{2} x_2.$$

Of course, this ought to be the formula for all integers k, positive or negative. To see this, we compute

$$0 = \mathrm{nm}(0) = \mathrm{nm}((-k) + k) = \mathrm{nm}(-k) + \mathrm{nm}(k) + \mathrm{tr}(-k^2) = \mathrm{nm}(-k) + k + \frac{k^2 - k}{2} x_2 - k^2 x_2$$

which gives

$$\mathrm{nm}(-k) = -k + (k^2 - \frac{k^2 - k}{2}) x_2 = -k + \frac{k^2 + k}{2} x_2 = (-k) + \frac{(-k)^2 + (-k)}{2} x_2,$$

as desired.

Next, we compute $\mathrm{nm}(t)$ directly. This is the class of the S_3-set $\mathrm{Map}_{C_3}(S_3, C_3/e)$. We see directly that this is an S_3-set of cardinality 9 with no C_3-fixed points, so it must be $[S_3/e] + [S_3/C_2] = x_3 x_2 + x_3$.

Now using our Mazur sum formula again, we get an explicit formula for $\mathrm{nm}_{C_3}^{S_3}$. Letting $a, b \in \mathbb{Z}$, we have:

$$\begin{aligned} \mathrm{nm}_{C_3}^{S_3}(a + bt) = \mathrm{nm}_{C_3}^{S_3}(a) + \mathrm{nm}_{C_3}^{S_3}(bt) + \mathrm{tr}_{C_3}^{S_3}(abt) &= a + \frac{a^2 - a}{2} x_2 + \left(b + \frac{b^2 - b}{2} x_2 \right) (x_3 x_2 + x_3) + ab x_3 x_2 \\ &= \boxed{a + \frac{a^2 - a}{2} x_2 + b x_3 + (ab + \frac{3b^2 - b}{2}) x_3 x_2}. \end{aligned}$$

Bingo!

References

[1] Kristen Luise Mazur. "On the Structure of Mackey Functors and Tambara Functors". ISBN: 9781303458828. PhD thesis. United States – Virginia: University of Virginia. 106 pp. URL: https://www.proquest.com/docview/1445384784/abstract/4CF83495B6304024PQ/1.